Book Team
Editor *Terry Wesner*
Developmental Editor *Terry Baker*
Production Editor *Mary Klein*

Bernard J. Klein Publishing

President *Joanne Klein*
Chief Executive Officer *Mary Ann Klein-Wesner*
Vice President, Publisher *Timothy J. Wesner*
Vice President, Director of Sales and Marketing *Robert S. Wesner*
Vice President, Director of Production *Thomas S. Wesner*

Cover credit: Terry Wesner

Production, illustrations, photo research, and composition by Publication Services, Inc.

GetMath Educational Software http://www.getmath.com

ISBN 1-932661-68-9

Printed in the United States of America by Bernard J. Klein Publishing

10 9 8 7 6 5 4

To Thomas Stewart Wesner

Dad

To Margot

Phil

GMAC Bank

Student Loans
for up to $40,000 per year *

Defer payments until after graduation.**
Fast preliminary approval, usually in minutes.

Apply Now!
Go to gmacbankfunding.com

Apply online in as little as 15 minutes

- Loans up to $40,000 per academic year*
- Good for tuition and other educational expenses: books, fees, a laptop, room and board, travel home, etc.
- Get a check in as few as 5 business days
- Start payments now or up to six months after graduation**
- Reduce your interest rate by as much as 0.50% with automatic payments***

All loans are subject to application and credit approval.

* Undergraduate and graduate borrowers may borrow annually up to the lesser of the cost of attendance or $30,000 ($40,000 for certain schools where it has been determined that the annual cost of attendance exceeds $30,000). Borrowers in the Continuing Education loan program may borrow annually up to $30,000.

** Undergraduate students may choose to defer repayment until six months after graduation or ceasing to be enrolled at least half time in school. Interest only and immediate repayment options also available.

*** A 0.25% interest rate reduction is available for borrowers who elect to have monthly principal and interest payments transferred electronically from a savings or checking account. The interest rate reduction will begin when automatic principal and interest payments start, and will remain in effect as long as automatic payments continue without interruption. This reduced interest rate will return to contract rate if automatic payments are cancelled, rejected or returned for any reason. Upon request, borrowers are also entitled to an additional 0.25% interest rate reduction if (1) the first 36 payments of principal and interest are paid on time, and (2) at any time prior to the 36th on time payment, the borrower who receives the monthly bill elects to have monthly principal and interest payments transferred electronically from a savings or checking account, and continues to make such automatic payments through the 36th payment. This reduced interest rate will not be returned to contract rate if, after receiving the benefit, the borrower discontinues automatic electronic payment. The lender and servicer reserve the right to modify or discontinue borrower benefit programs (other than the co-signer release benefit) at any time without notice.

Love The Taste.
Taste The Love.

At Culver's® we can't think of anything better than serving up our creamy frozen custard and delicious classics cooked fresh the minute you order them. Which is why when we bring them to your table, they're always accompanied by a warm smile and a friendly offer to see if there's anything else we can get for you. So come on into your neighborhood Culver's and see for yourself. You might just be in love by the time you leave.

www.culvers.com

Contents

C H A P T E R 5

The Trigonometric Functions

C H A P T E R 6

Radian Measure, Properties of the Trigonometric and Inverse Trigonometric Functions

C H A P T E R 7

Trigonometric Equations

C H A P T E R 8

Additional Topics in Trigonometry

C H A P T E R 9

Exponential and Logarithmic Functions

Chapter 10 Systems of Linear Equations and Inequalities

Chapter 11 The Conic Sections

Chapter 12 Topics in Discrete Mathematics

College Loan Corporation

CLC

Need more money for college expenses?

The CLC Private Loan℠ can get you up to **$40,000** a year for college-related expenses.

Here's why the CLC Private Loan℠ is a smart choice:

- ☑ Approved borrowers are sent a check within four business days
- ☑ Get $1000 - $40,000 each year
- ☑ Interest rates as low as prime + 0% (8.66% APR)
- ☑ Quick and easy approval process
- ☑ No payments until after graduating or leaving school

CLICK HERE

or call **800.311.9615.**

We are available 24 hours a day, 7 days a week.

8.66% APR is based on a Prime Rate of 8.25% as of August 1, 2006.

C H A P T E R 12

Topics in Discrete Mathematics

In this chapter we investigate several topics that have wide application in science and business as well as in advanced mathematics. The term **discrete mathematics** refers to the fact that most of these topics can be discussed largely in terms of integers. Discrete is here used in the sense of "distinct." Discrete mathematics has taken on new importance with the advent of electronic digital computers.

12–1 Sequences

A new employee is hired at $26,000 and is told to expect an 8% raise each year for the first six years. What is the employee's pay in each year from the first to the sixth?

This problem describes what is called a series. In this section we study the mathematics that can deal with series, and therefore that can deal with this type of problem.

General principles

A **sequence** is a list of numbers; for example, 1, 3, 5, 7, 9, . . . is a sequence. The ellipsis (. . .) indicates the list goes on indefinitely, making this an infinite sequence. The numbers in the list are called **terms.** Since there is a first, second, third, etc. term in the list, the terms can be paired up with the positive integers.

Sequence

A **finite sequence** is a list of numbers that can be paired up with the set of positive integers 1, 2, 3,. . . , n for some positive integer n.

An **infinite sequence** is a list of numbers that can be paired up with the set of positive integers 1, 2, 3, ...

We refer to the values 1, 2, 3, . . . , n as the **domain** of the sequence. In general, *we define a sequence by a formula for its nth term.* If we call the sequence A, the nth term is called a_n. If the sequence is called B, we call the nth term b_n, etc. Example 12–1 A illustrates this.

■ Example 12–1 A

1. An infinite sequence is defined by the expression $a_n = 3n + 2$. List the first three terms of the sequence.

$$\begin{aligned} a_n &= 3n + 2 \\ a_1 &= 3(1) + 2 = 5 \\ a_2 &= 3(2) + 2 = 8 \\ a_3 &= 3(3) + 2 = 11 \end{aligned}$$

Thus, the sequence looks like 5, 8, 11, . . . $3n + 2$, ...

2. List the first four terms of the sequence in which the nth term is $a_n = (-1)^n(2^n - n)$.

$$\begin{aligned} a_1 &= (-1)^1(2^1 - 1) = -1 \\ a_2 &= (-1)^2(2^2 - 2) = 2 \\ a_3 &= (-1)^3(2^3 - 3) = -5 \\ a_4 &= (-1)^4(2^4 - 4) = 12 \end{aligned}$$

Thus, the sequence looks like -1, 2, -5, 12, . . . $(-1)^n(2^n - n)$. . . . ■

There are times when we wish to find an expression for the general term a_n. This is not always easy or even possible, but in many instances examination of the values of the terms will provide some guidance. This is illustrated in example 12–1 B.

■ Example 12–1 B

Find an expression for the general term of each sequence.

1. 7, 12, 17, 22, 27, . . .
   Subtract 2 from each term, giving the sequence 5, 10, 15, 20, . . . in which the nth term is of the form $5n$. The general term of the original sequence is 2 more than this: $a_n = 5n + 2$.

2. -3, 9, -15, 21, . . .
   The alternating signs can be accounted for by a factor of $(-1)^n$. We thus consider the positive valued sequence 3, 9, 15, 21, ...
   Subtracting 3 from each term yields the sequence 0, 6, 12, 18, . . . in which we have multiples of 6, with general term $6(n - 1)$.
   Thus, the sequence 3, 9, 15, 21, . . . can be expressed by the general term $6(n - 1) + 3 = 6n - 3$, and the general term of the original sequence is $a_n = (-1)^n(6n - 3)$.

3. A manufacturer is testing an electronics device in a high-heat situation. The total number of failed devices is recorded each hour. In the first 5 hours, the total numbers of failed devices recorded each hour are 3, 8, 12, 18, and 23. Create a sequence that approximates this pattern and use this to predict the number of failed devices that will be counted in the seventh hour.

If we make a list of the *increase* in the number of failed devices each hour we obtain the sequence 5, 4, 6, 5. Note that the average of these values is 5. It would seem logical to simulate the actual sequence with the following one: 3, 8, 13, 18, 23, in which the increase is a constant, 5. Using this sequence, we obtain a sixth element of 28 and a seventh element of 33. Thus, it might be reasonable to predict that the number of failed devices that will be counted in the seventh hour will be 33. The general term for this sequence is $a_n = 5n - 2$. ■

Arithmetic sequences

Many sequences encountered in practice are a variety in which each succeeding element can be found by adding a constant to the previous term. For example, if a_1 is 5, and we add 3 to get each next term, then the sequence is 5, 8, 11, 14, ... These sequences are called **arithmetic sequences.**

> **Arithmetic sequence**
> An arithmetic sequence A is a sequence in which $a_{n+1} = a_n + d$ for all terms of the sequence and for some (fixed) real number d.

Observe that the definition implies that $a_{n+1} - a_n = d$ for all terms of the sequence. In other words the difference between successive terms is a constant. We call d the **common difference.**

■ ***Example 12–1 C***

1. 40, 34, 28, 22, 16 is an arithmetic sequence, with $d = -6$.

2. 2, 5, 9, 14, 20 is not an arithmetic sequence since the difference between terms is not constant. ■

The general term of an arithmetic sequence

To find a general expression for any term a_n of an arithmetic sequence consider the following pattern.

$$
\begin{aligned}
&a_1\\
&a_2 = a_1 + d\\
&a_3 = a_1 + 2d\\
&a_4 = a_1 + 3d
\end{aligned}
$$

This suggests the following.

> **General term of an arithmetic sequence**
> If A is an arithmetic sequence with first term a_1 and common difference d, then the general term a_n is
> $$a_n = a_1 + (n - 1)d$$

Example 12–1 D illustrates some uses of this formula for the general term of an arithmetic sequence.

■ ***Example 12–1 D***

1. Given an arithmetic sequence in which $a_1 = 100$ and $a_{21} = 10$. Find a_{50}.
We must find d to apply the formula. The fact $a_{21} = 10$ gives us a value for n and for a_n to substitute into the general formula; we also use the value 100 for a_1.

$a_n = a_1 + (n - 1)d$ General term of an arithmetic sequence
$a_{21} = a_1 + (21 - 1)d$ General term when $n = 21$
$10 = 100 + (21 - 1)d$ Replace a_{21} with 10, a_1 with 100
$-\frac{9}{2} = d$

We can now find a_{50}.

$a_{50} = 100 + 49(-\frac{9}{2})$ $a = 100$, $d = -\frac{9}{2}$, $n = 50$
$= -120\frac{1}{2}$

2. Find the number of terms in the arithmetic sequence $-9, -4, 1, 6, \ldots, 111$.
We can determine that $a_1 = -9$ and $d = 5$ from the first few terms. We know that 111 is the nth term; we just don't know n yet.

$$a_n = a_1 + (n - 1)d$$
$$111 = -9 + (n - 1)(5)$$
$$25 = n$$

Thus, 111 is the 25th term, so the sequence has 25 terms. ■

Geometric sequences

Another common type of sequence is called a **geometric sequence.** Geometric sequences are used to describe the growth of everything from populations to bank accounts.

In a geometric sequence, each next element can be found by multiplying the previous term by a constant, called r, for **common ratio.** For example, if a_1 is 5, and the common ratio is 2, then the sequence is

$$\begin{array}{lllll} 5, & 5\cdot 2, & 5\cdot 2\cdot 2, & 5\cdot 2\cdot 2\cdot 2, \ldots & \\ 5, & 5\cdot 2, & 5\cdot 2^2, & 5\cdot 2^3, \ldots & \\ 5, & 10, & 20, & 40, & 80, \ldots \end{array}$$

> **Geometric sequence**
> A geometric sequence is a sequence in which $a_{n+1} = r \cdot a_n$ for all terms of the sequence and for some real number r. We require $r \neq 0$ and $a_1 \neq 0$.

The restrictions on r and a_1 are so that any given geometric sequence has only one description in terms of a_1 and r. See problem 91 in the exercises for an illustration of what happens without these restrictions.

Observe that the definition implies that $\frac{a_{n+1}}{a_n} = r$ for all terms of the sequence. This can be used to determine if a sequence is or is not a geometric sequence.

■ ***Example 12–1 E***

1. 1, −2, 4, −8, 16 is a geometric sequence, with $r = -2$, because

$$\frac{-2}{1} = \frac{4}{-2} = \frac{-8}{4} = \frac{16}{-8} = -2$$

2. 2, 4, 6, 8 is not a geometric sequence since the ratio of successive terms is not constant. (It is in fact an arithmetic sequence.) Observe that $\frac{4}{2} = 2$ but $\frac{6}{4} = 1.5$. ■

The general term of a geometric sequence

To find a general expression for the nth term a_n of a geometric sequence consider the expressions

$$\begin{aligned} &a_1 \\ a_2 &= a_1 r \\ a_3 &= a_2 r = (a_1 r)r = a_1 r^2 \\ a_4 &= a_3 r = (a_1 r^2)r = a_1 r^3 \end{aligned}$$

which suggest the following.

> **General term of a geometric sequence**
> If A is a geometric sequence with first term a_1 and common ratio r, then the general term a_n is
> $$a_n = a_1 r^{n-1}$$

The use of the general term of a geometric sequence is shown in example 12–1 F.

■ ***Example 12–1 F***

1. Find the fourth term of a geometric sequence in which $a_1 = 5$ and $r = 1\frac{1}{4}$.

$$a_4 = a_1 r^3 = 5\left(\frac{5}{4}\right)^3 = 5\left(\frac{5^3}{4^3}\right) = \frac{5^4}{4^3} = \frac{625}{64}$$

2. Given a geometric sequence in which $a_1 = 160$ and $a_8 = \frac{5}{4}$, find a_4.

$a_n = a_1 r^{n-1}$	General formula
$a_8 = a_1 r^{8-1}$	General formula for $n = 8$
$\frac{5}{4} = 160r^7$	Replace a_8 with $\frac{5}{4}$ and a_1 with 160
$\frac{5}{4} \cdot \frac{1}{160} = \frac{1}{160} \cdot 160r^7$	Multiply each member with $\frac{1}{160}$
$\frac{1}{128} = r^7$	
$(\frac{1}{2})^7 = r^7$	
$\frac{1}{2} = r$	

We can now compute a_4.

$$a_4 = a_1 r^3 = 160(\tfrac{1}{2})^3 = 160(\tfrac{1}{8}) = 20$$ ■

Notes on the general expression of a sequence

It needs to be noted that the first few terms of a sequence do not necessarily give enough information to find the general term; consider the sequence 1, 2, 4, ... One expression for a_n is 2^{n-1}. This general term produces 8 for the fourth term. However, the expression $a_n = \frac{1}{2}n^2 - \frac{1}{2}n + 1$ also produces 1, 2, 4 for the first three terms, but 7 for the fourth term. In fact, given the first n terms of a sequence it is possible to derive an unlimited number of expressions for the general term; we will not pursue this further here,[1] however, except to say that to fully specify a sequence we must actually include a rule or expression for the general term.

Further, an expression does not always suffice for the general term. For some sequences, no expression is possible. For example, we could specify a sequence with the *rule* that the nth term is the nth digit in the decimal expansion for π. Thus, the sequence looks like 3, 1, 4, 1, 5, 9, 2, ... It is impossible (even in theory) to find a general expression for the nth term of this sequence.

Mastery points

Can you

- Find the terms of a sequence, given the general term?
- Find any given term of a sequence from the general term?
- Find an expression for the general term of a sequence, given the first few terms of the sequence?
- Identify arithmetic and geometric sequences?

Exercise 12–1

List the first four terms of each sequence.

1. $a_n = \frac{5}{2}n - 3$ **2.** $a_n = \dfrac{3n - 4}{2}$ **3.** $b_n = 2^n - n^2$ **4.** $b_n = (\frac{1}{2})^n$

5. $a_n = 3$ **6.** $b_n = 30 - n^2$ **7.** $c_n = n^2 - 4n + 2$ **8.** $a_n = (n - 5)^3$

9. $a_n = \dfrac{\sqrt{n}}{n + 1}$ **10.** $c_n = \dfrac{n}{n - 2}$ **11.** $b_n = (n - 2)(n + 3)$ **12.** $a_n = 11 + \dfrac{n}{5}$

Find an expression for the general term of each sequence.

13. 2, 5, 8, 11, . . . **14.** 3, 6, 9, 12, . . . **15.** $-20, -16, -12, \ldots$ **16.** $-8, -3, 2, 7, \ldots$

17. $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4} \ldots$ **18.** $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ **19.** $\frac{1}{6}, \frac{2}{11}, \frac{3}{16}, \frac{4}{21}, \ldots$ **20.** $\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \ldots$

21. $\frac{1}{4}, -\frac{4}{9}, \frac{9}{16}, -\frac{16}{25}, \ldots$ **22.** 2, 6, 12, 20, . . . **23.** $1, -1, 1, -1, \ldots$ **24.** 1, 0.1, 0.01, 0.001, . . .

[1]See the last two problems in the exercises for further development.

25. A biology researcher measured the population of a certain insect under laboratory conditions every 5 hours and obtained the values 300, 400, 530, 710. Approximately what value might the researcher expect for the next measurement?

26. A manufacturer makes integrated circuits by putting a certain number of circuits on a circular silicon wafer. The manufacturer can vary the number of circuits per wafer by varying the diameter of the wafer. The manufacturer has found that the number of bad circuits per wafer varies in the manner shown in the table. How many bad circuits would the manufacturer expect on a wafer with diameter 8″?

Diameter	Number of bad circuits
2″	4
3″	9
4″	16
5″	25

Characterize the sequence from the problem indicated as arithmetic, geometric, or neither. State the common difference or common ratio as appropriate.

27. Problem 1 **28.** Problem 2 **29.** Problem 3 30. Problem 4 **31.** Problem 5
32. Problem 6 **33.** Problem 7 **34.** Problem 8 35. Problem 9 **36.** Problem 10
37. Problem 11 **38.** Problem 12 **39.** Problem 13 **40.** Problem 14 41. Problem 15
42. Problem 16 **43.** Problem 17 **44.** Problem 18 **45.** Problem 19 **46.** Problem 20
47. Problem 21 **48.** Problem 22 **49.** Problem 23 **50.** Problem 24

Solve the following problems.

51. Find the 12th term of an arithmetic sequence in which $a_1 = 6$ and $d = 5$.

52. Given an arithmetic sequence in which $a_1 = 10$ and $a_{21} = 220$, find a_{25}.

53. Find the number of terms in the arithmetic sequence 7, 11, 15, . . . , 135.

54. Find the 15th term of an arithmetic sequence in which $a_1 = -200$ and $d = 3$.

55. Given an arithmetic sequence in which $a_1 = -40$ and $a_{15} = 40$, find a_{14}.

56. Find the number of terms in the arithmetic sequence 150, $148\frac{1}{2}$, 147, . . . , 39.

57. Find the 27th term of an arithmetic sequence in which $a_1 = 6$ and $a_{40} = 300$.

58. Given an arithmetic sequence in which $a_1 = -46$ and $a_{21} = 150$, find a_{11}.

59. Find the number of terms in the arithmetic sequence $-42, -39\frac{1}{2}, \ldots, -9\frac{1}{2}$.

60. Find a_{200} of the arithmetic sequence where $a_1 = 19$ and $a_{10} = 28$.

61. In an arithmetic sequence, $a_{15} = 49$ and $a_{28} = 88$. Find a_4.

Find a_n for the following geometric sequences for the given values of a_1, r, and n.

62. $a_1 = 10, r = -2, n = 5$ **63.** $a_1 = \frac{1}{4}, r = -2, n = 6$ **64.** $a_1 = \frac{1}{40}, r = 4, n = 4$
65. $a_1 = -12, r = -\frac{1}{2}, n = 5$ **66.** $a_1 = 400, r = 0.1, n = 3$ **67.** $a_1 = \frac{2}{3}, r = \frac{3}{2}, n = 3$

68. Find the 4th term of a geometric sequence in which $a_1 = 4$ and $a_2 = 1$.

69. Given a geometric sequence in which $a_1 = 5$ and $a_5 = \frac{5}{81}$, find a_4.

70. Given a geometric sequence in which $a_3 = \frac{1}{3}$ and $a_6 = -\frac{1}{81}$, find a_2.

71. Find the sixth term of a geometric sequence in which $a_1 = \frac{9}{25}$ and $r = 5$.

72. Given a geometric sequence in which $a_1 = \frac{1}{32}$ and $a_4 = -\frac{1}{4}$, find a_5.

73. Given a geometric sequence in which $a_2 = \frac{5}{4}$ and $a_4 = 5$, find a_1.

74. A pendulum swings a distance of 20 inches on its first swing. Each subsequent swing is 95% of the previous distance. How far does the pendulum swing on the (a) fourth swing, and (b) eighth swing?

75. A new employee is hired at \$26,000 and is told to expect an 8% raise each year for the first six years. What is the employee's pay in each year from the first to the sixth?

76. A ball is dropped from a height of 10 feet. If the ball rebounds three-fourths of the height of its previous fall with each bounce, how high does it rebound on the (a) third bounce, (b) sixth bounce, and (c) nth bounce?

77. A machine is to be depreciated by what is called the "constant percentage method." A certain, fixed percentage will be deducted from the machine's value every year. Suppose the machine cost \$15,000 and has a scrap value of \$3,000 after six years. Find the rate r of depreciation. That is, solve $15{,}000\,(1 - r)^6 = 3{,}000$ for r. Find r to the nearest 0.1%.

Problems 78 through 83 are related. In problems 78, 79, and 80, assume two arithmetic sequences a and b such that $a_n = 2n + 5$ and $b_n = 1 - n$.

78. Define a new sequence c such that $c_n = a_n + b_n$.
a. Write the first four terms of the sequences a, b, and c. **b.** Is the sequence c arithmetic?
c. Find an expression for c_n and use it to compute c_{20}.

79. Define a new sequence d such that $d_n = 3a_n$.
a. Write the first four terms of the sequences a and d.
b. Is d an arithmetic sequence?
c. Find an expression for d_n and use it to compute d_{20}.

80. Define a new sequence e such that $e_n = a_n \cdot b_n$.
a. Write the first four terms of the sequences a, b, and e. **b.** Is e an arithmetic sequence?
c. Find an expression for e_n and use it to compute e_{20}.

81. Suppose a_n and b_n are two arithmetic sequences, and a new sequence c is defined such that $c_n = a_n + b_n$. Is the new sequence an arithmetic sequence? Prove or disprove this statement.

82. Suppose a_n is an arithmetic sequence and k is some constant. Is the sequence b defined as $b_n = ka_n$ an arithmetic sequence? Prove or disprove this statement.

83. Suppose a_n and b_n are two arithmetic sequences, and a new sequence c is defined such that $c_n = (a_n)(b_n)$. Is the new sequence an arithmetic sequence? Prove or disprove this statement.

Problems 84 through 89 are related. In problems 84, 85, and 86 assume two geometric sequences a and b such that $a_n = 3^n$ and $b_n = 3(2^n)$.

84. Define a new sequence c such that $c_n = a_n + b_n$.
a. Write the first four terms of the sequences a, b, and c. **b.** Is the sequence c geometric?
c. Find an expression for c_n and use it to compute c_5.

85. Define a new sequence d such that $d_n = \frac{1}{2}a_n$.
a. Write the first four terms of the sequences a and d.
b. Is d a geometric sequence?
c. Find an expression for d_n and use it to compute d_5.

86. Define a new sequence e such that $e_n = a_n \cdot b_n$.
a. Write the first four terms of the sequences a, b, and e. **b.** Is e a geometric sequence?
c. Find an expression for e_n and use it to compute e_5.

87. Suppose a_n and b_n are two geometric sequences, and a new sequence c is defined such that $c_n = a_n + b_n$. Is the new sequence a geometric sequence? Prove or disprove this statement.

88. Suppose a_n is a geometric sequence and k is some constant. Is the sequence b defined as $b_n = ka_n$ a geometric sequence? Prove or disprove this statement.

89. Suppose a_n and b_n are two geometric sequences, and a new sequence c is defined such that $c_n = (a_n)(b_n)$. Is the new sequence a geometric sequence? Prove or disprove this statement.

90. A store charges \$5 to develop a roll of film. With each roll of film the customer gets a coupon. With four coupons the customer gets a roll developed free. Let A be the sequence in which a_n represents the *average* price for developing n rolls of film, one after the other. Find an expression for a_n.

Hint: The greatest integer function may be helpful. This is the function $f(x) = [x]$, in which $[x]$ is the greatest integer less than or equal to x. For $x > 0$ this is equivalent to throwing away the fractional part of a number. For example, $[1.8] = 1$, $[9\frac{1}{2}] = 9$, etc.

91. In the definition of a geometric sequence we required $r \neq 0$ and $a_1 \neq 0$. If we remove these restrictions, which of the following would be a geometric sequence?
a. 5, 0, 0, 0, 0, . . .
b. −3, 0, 0, 0, . . .
c. 0, 0, 0, 0, . . .
d. $a_1 = \sqrt{2}$, $r = 0$
e. $a_1 = 0$, $r = \sqrt{2}$
f. $a_1 = 0$, $r = -25{,}000$

92. The following is sometimes called the "Coxeter-Ulam" algorithm.
First, select any natural number as the first element of the sequence.
Perform the following procedure to get the next element of the sequence.
- If the element is even, divide by 2.
- If the element is odd, multiply by 3 and add 1.

Continue this procedure to derive new elements in the sequence, but stop if the element is one.
This algorithm will generate a sequence of numbers and each sequence seems to have a certain property in common. Generate a few of these sequences and determine this property. (No one has been able to prove that all such sequences have this property.) (*Hint:* Start with a first element of 2, 3, 4, 5, 6. Then try 21, then 7, as the first element. Try other values at random.)

As stated in the text, given the first, say, 3 terms of a sequence (or the first n terms, for that matter), it is possible to find an unlimited number of expressions for the general term. One way to provide some more examples is to take any geometric sequence with terms a_n, and assume there is some other sequence b_n that has the same three first elements. Substitute the pairs of values $(1,b_1)$, $(2,b_2)$, and $(3,b_3)$ into $b_n = An^2 + Bn + C$. (This is called finding a "quadratic interpolation formula.")

For example, consider the geometric sequence 2, 3, $4\frac{1}{2}$, with $a_1 = 2$, $r = \frac{3}{2}$. Consider these as ordered pairs $(n,b_n) = (1,2), (2,3), (3,4\frac{1}{2})$ and substitute.

$$b_n = An^2 + Bn + C$$

$$\begin{aligned}
n = 1&: \quad 2 = A(1^2) + B(1) + C;\ 2 = A + B + C \\
n = 2&: \quad 3 = A(2^2) + B(2) + C;\ 3 = 4A + 2B + C \\
n = 3&: \quad 4\tfrac{1}{2} = A(3^2) + B(3) + C;\ 4\tfrac{1}{2} = 9A + 3B + C
\end{aligned}$$

Now, solve this system of three equations in three unknowns (chapter 10) for A, B, and C. We obtain $A = B = \frac{1}{4}$, $C = \frac{3}{2}$ so the expression is $b_n = \frac{1}{4}n^2 + \frac{1}{4}n + \frac{3}{2}$.

In the geometric sequence, the formula is $a_n = 2(\frac{3}{2})^{n-1}$, and $a_4 = 6\frac{3}{4}$, whereas $b_4 = 6\frac{1}{2}$. Thus, given the three terms 2, 3, $4\frac{1}{2}$, we can find at least two different sequences that begin with these terms.

93. Using this example, find a quadratic expression that defines a sequence that begins with the same three terms as the given sequence but is different in the fourth term.
a. geometric sequence with $a_1 = 9$, $r = \frac{2}{3}$
b. geometric sequence with $a_1 = 3$, $r = \frac{2}{3}$
c. 3, 1, 4, 1, . . . , where a_n is the nth term in the decimal expansion of π

94. Ramsey theory, named for Frank Plumpton Ramsey, an English mathematician in the first half of the twentieth century, discusses finding order in disorder. One unexpected implication of this theory is the following.

Take the arithmetic progression 1, 2, 3, 4, 5, 6, 7, 8, 9. Underline some of the values, and leave the rest not underlined. Ramsey theory indicates that either three of the underlined or three of the not-underlined values will form an arithmetic progression.[2] For example, consider the arrangement

$$1 \quad \underline{2} \quad 3 \quad \underline{4} \quad 5 \quad 6 \quad \underline{7} \quad 8 \quad 9$$

The not-underlined values 1, 3, and 5 form an arithmetic progression. In

$$1 \quad \underline{2} \quad 3 \quad \underline{4} \quad \underline{5} \quad 6 \quad \underline{7} \quad 8 \quad 9$$

the values 3, 6, and 9 form an arithmetic progression.

Create two other arrangements by underlining some (or none) of the values in the sequence 1 . . . 9, then find at least one arithmetic progression in each.

[2]For an understandable, short proof of this fact see the excellent article "Ramsey Theory" by Ronald L. Graham and Joel H. Spencer in the July 1990 issue of *Scientific American* magazine.

Skill and review

1. If $3 + 6 + 9 + \cdots + 3n = 231$, what is $1 + 2 + 3 + \cdots + n$?
2. Find the sum $(1 - 5) + (5 - 9) + (9 - 13) + \cdots + (81 - 85)$.
3. If $x_1 + x_2 + x_3 + \cdots + x_n = 420$, what is $3x_1 + 3x_2 + 3x_3 + \cdots + 3x_n$?
4. If $a_1 + a_2 + a_3 + \cdots + a_n = 500$ and $b_1 + b_2 + b_3 + \cdots + b_n = 200$, what is $(a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n)$?
5. Graph $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$.
6. Solve $|3 - \frac{1}{2}x| > 12$.

Extra Credit Rocks

Sign up for a Discover® Student Card today and enjoy:

- 0% Intro APR* on Purchases for 6 Months
- No Annual Fee
- Easiest Online Account Management Options
- Full 5% *Cashback Bonus*®* on Get More purchases in popular categories all year
- Up to 1% *Cashback Bonus*®* on all your other purchases
- Unlimited cash rewards that never expire as long as you use your Card

APPLY NOW

DISCOVER®
CARD

*View Discover® Card Rates, Fees, Rewards and Other Important Information.

12–2 Series

A vacuum pump removes one-fifth of the remaining air in a tank with each stroke. (a) How much air remains in the tank after the fifth stroke? (b) How many strokes would be necessary to remove 98% of the air?

The solution to the problem posed here involves the mathematics of series, which is what we study in this section.

An expression that indicates the summation of the terms of a sequence is called a **series.** Such a sum might represent the total distance traveled by an accelerating object, the population of a town after a few years, or the present value of an annuity that will pay for someone's education.

For the finite sequence

$$5, 10, 15, 20$$

the expression

$$5 + 10 + 15 + 20$$

is its series. The resulting value, 50, is the sum of the series. We would also refer to 5, 10, 15, and 20 as the terms of the series.

Sigma notation

Sigma notation[3] is a convenient way to express a series in more compact form. Σ is the capital letter *sigma* in the Greek alphabet. We use it to indicate the word "sum," or "series." For example, $\sum_{i=1}^{4}(3i + 2)$ is read "the sum of the terms $3i + 2$ as i takes on the integer values from 1 through 4." It means that the ith term of a series is $3i + 2$, and that we are interested in terms 1 through 4, giving the series

i	1	2	3	4
term	$(3 \cdot 1 + 2) +$	$(3 \cdot 2 + 2) +$	$(3 \cdot 3 + 2) +$	$(3 \cdot 4 + 2)$
series		$5 + 8 + 11 + 14$		

Note that the letter i was not significant in the notation; we could have used any letter we wished. Example 12–2 A illustrates expanding sigma notation.

■ **Example 12–2 A**

Expand the following sigma expressions.

1. $\sum_{j=3}^{6}(4j - 1)$

$(4 \cdot 3 - 1) + (4 \cdot 4 - 1) + (4 \cdot 5 - 1) + (4 \cdot 6 - 1)$

$11 + 15 + 19 + 23$

[3]The Σ symbol was first used by Leonhard Euler in 1755.

2. $\sum_{k=2}^{5} (-1)^k \left(\frac{k}{k-1}\right)^2$

$(-1)^2 \left(\frac{2}{1}\right)^2 + (-1)^3 \left(\frac{3}{2}\right)^2 + (-1)^4 \left(\frac{4}{3}\right)^2 + (-1)^5 \left(\frac{5}{4}\right)^2$

$4 - \frac{9}{4} + \frac{16}{9} - \frac{25}{16}$ ■

If the number of terms in the series is infinite, the corresponding series may not have a sum. For example, if the sequence is 1, 3, 5, 7, . . . , the series is $1 + 3 + 5 + 7 + . . .$, which cannot be summed. We can, however, add up finite subsequences. For example,

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16, \text{ etc.}$$

Sum of the terms of a finite arithmetic sequence

To obtain an expression for the sum of the first n terms (S_n) of an arithmetic sequence we write the expression for the sum forward and also backward, as shown, and add up the terms on each side of the equations.

$$\begin{aligned}
S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n-1)d] \\
S_n &= a_n + (a_n - d) + (a_n - 2d) + \cdots + [a_n - (n-1)d] \\
\hline
2S_n &= \underbrace{(a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n)}_{n \text{ terms of } (a_1 + a_n)}
\end{aligned}$$

$$2S_n = n(a_1 + a_n)$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Sometimes we do not know the value of a_n; we can derive a formula for this purpose.

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum of first n terms of an arithmetic series

$a_n = a_1 + (n - 1)d$ Expression for a_n

$S_n = \frac{n}{2}[a_1 + (a_1 + (n - 1)d)]$ Replace a_n by $a_1 + (n - 1)d$

$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ Formula for S_n.

This formula does not require that a_n be calculated to find S_n. This is summarized as follows.

Sum of the first n terms of an arithmetic sequence

The sum S_n of the first n terms of an arithmetic sequence with first term a_1 and nth term a_n is

$$S_n = \frac{n}{2}(a_1 + a_n)$$

or

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

S_n is called the nth **partial sum.**

Example 12–2 B illustrates summing finite arithmetic sequences.

■ *Example 12–2 B*

Find the required sum.

1. $a_n = 2n - 1$; sum the first 5 terms.
Compute a_1 and a_5 first.

$a_1 = 1;\ a_5 = 9$ — Use $a_n = a_1 + (n - 1)d$

$S_5 = \frac{5}{2}(1 + 9) = 25$ — Use $S_n = \frac{n}{2}(a_1 + a_n)$; $n = 5$, $a_1 = 1$, $a_5 = 9$

2. Add up the values 2, 9, 16, 23, . . . , 65.
$a_1 = 2$, $d = 7$. We need to know how many terms there are.

$a_n = a_1 + (n - 1)d$ — General expression for nth term

$65 = 2 + (n - 1)(7)$ — Find n for which $a_n = 65$

$10 = n$

Thus, 65 is a_{10}, and we therefore want to find S_{10}.
$S_{10} = \frac{10}{2}(2 + 65) = 5(67) = 335$ ■

Sum of the terms of a finite geometric sequence

Just as the expression just discussed determines the nth partial sum of an arithmetic sequence, there is an expression that determines the nth partial sum of a geometric sequence. The derivation of this formula is left as an exercise.

Sum of the first n terms of a geometric sequence

The nth partial sum S_n of the first n terms of a geometric sequence with first term a_1 and ratio r, $r \neq 1$, is

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

S_n is often called the nth **partial sum.**

Example 12–2 C

Find the required nth partial sum.

1. A geometric series with $a_1 = 3$, $r = \frac{1}{2}$, and $n = 6$.

$$S_6 = \frac{3(1 - (\frac{1}{2})^6)}{1 - \frac{1}{2}} = 5\tfrac{29}{32}$$

2. $\sum_{k=1}^{7} 3(2)^k$

The series is $3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + \cdots + 3 \cdot 2^7$. Thus $a_1 = 6$ and $r = 2$. We want S_7.

$$S_7 = \frac{6(1 - 2^7)}{1 - 2} = 762$$

Infinite geometric series

Suppose a sprinter runs a 100-meter race. One view of the race is as follows. First, the runner runs to the 50-meter position, or half the distance to the finish ($\frac{1}{2}$ of the total distance). Then, the runner runs to the 75-meter position, or half the remaining distance ($\frac{1}{4}$ of the total distance). Next, the runner runs half the remaining distance, getting to the 87.5-meter mark ($\frac{1}{8}$ of the total distance). The runner goes on and on in this fashion, always attaining a goal and then running half the distance remaining to the finish. See figure 12–1.

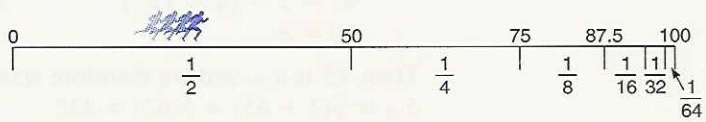


Figure 12–1

Note that the runner covers the following parts of the course: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$. Since the runner completes one race, it makes sense to say that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$.

Now the values $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots, \frac{1}{2^n}, \cdots$ determine a geometric series that does not terminate. Such a series is said to be an **infinite geometric series.** The nth partial sum would be $\sum_{i=1}^{n} (\frac{1}{2})^n = \frac{\frac{1}{2}(1 - (\frac{1}{2}))^n}{1 - \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n$. As n gets greater and greater, the fraction $(\frac{1}{2})^n$ in the expression $1 - (\frac{1}{2})^n$ gets closer and closer to 0, and so the nth partial sum gets closer and closer to the value 1. This is in accord with what we said about running the

race. In like fashion, the value of r^n gets less and less as n gets greater and greater in the expression $\frac{a_1(1 - r^n)}{1 - r}$, as long as $|r| < 1$. This means that the expression gets closer and closer to $\frac{a_1(1 - 0)}{1 - r} = \frac{a_1}{1 - r}$, leading to the following definition.

Sum of an infinite geometric series

If $|r| < 1$ the sum S of the terms of an infinite geometric series is

$$S = \frac{a_1}{1 - r}$$

Note If $|r| \geq 1$ the sum is not defined. To see why, consider the series $1 + 3 + 9 + 27 + \cdots + 3^{n-1} + \cdots$. As each term is added, this sum grows larger and larger, by ever increasing amounts.

Summing infinite geometric sequences is illustrated in example 12–2 D.

■ ***Example 12–2 D***

Find the sum of the infinite geometric series.

1. $\sum_{i=1}^{\infty} 3(\frac{2}{3})^i$

The symbol ∞ (infinity) means there is no last term. This is an infinite geometric series with $a_1 = 2$ and $r = \frac{2}{3}$, and, since $|r| < 1$, the infinite sum is defined. $S = \frac{2}{1 - \frac{2}{3}} = \frac{2}{\frac{1}{3}} = 6$.

2. $\sum_{i=1}^{\infty} (\frac{4}{3})^i$

$a_1 = \frac{4}{3}$ and $r = \frac{4}{3}$. Since $|r| > 1$ this series does not have a sum. ■

Rational form of repeating decimal numbers

Repeating decimal numbers can be viewed as the sums of infinite geometric series. This can be used to find their rational form. For example, we are familiar with the fact that $\frac{2}{3} = 0.66\overline{6}$, but what is $0.77\overline{7}$? Example 12–2 E answers this question.

■ ***Example 12–2 E***

Find the rational number form of the repeating decimal.

1. $0.77\overline{7}$

This is $\frac{7}{10} + \frac{7}{100} + \frac{7}{1{,}000} + \cdots$ or $\frac{7}{10} + 7\left(\frac{1}{10}\right)^2 + 7\left(\frac{1}{10}\right)^3 + \cdots$,

which is an infinite geometric series with $a_1 = 0.7$ and $r = 0.1$. Thus,

$S = \frac{a}{1 - r} = \frac{0.7}{1 - 0.1} = \frac{0.7}{0.9} = \frac{7}{9}$.

2. $0.343\overline{434}$

This can be written as $0.34 + 0.0034 + 0.000034 + \cdots$, or $34\left(\frac{1}{100}\right) + 34\left(\frac{1}{100}\right)^2 + 34\left(\frac{1}{100}\right)^3 + \cdots$, which is an infinite geometric series with $a_1 = 0.34$ and $r = 0.01$, so $S = \frac{0.34}{1 - 0.01} = \frac{0.34}{0.99} = \frac{34}{99}$.

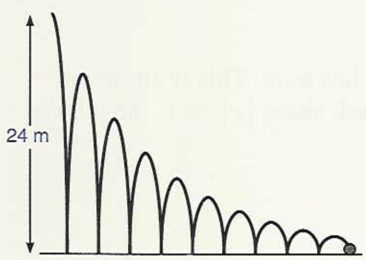
3. $0.53\overline{131}$

If $x = 0.53\overline{131}$, then $10x = 5.3\overline{131}$. We first find the value of $0.3\overline{131}$, which is an infinite geometric progression in which $a_1 = 0.31$ and $r = 0.01$, so $S = \frac{0.31}{1 - 0.01} = \frac{0.31}{0.99} = \frac{31}{99}$.

Thus, $10x = 5\frac{31}{99} = \frac{526}{99}$, so $x = \frac{1}{10}\left(\frac{526}{99}\right) = \frac{263}{495}$.

Infinite geometric series find wide application in solving problems from areas outside of mathematics.

■ Example 12–2 F



A ball is dropped from a height of 24 meters. Each time it strikes the floor, the ball rebounds to a height that is three-fourths of the previous height. Find the total vertical distance that the ball travels before it comes to rest on the floor. See the figure.

The sequence of distances that the ball travels would be 24, <u>18</u>, 18, <u>13.5</u>, 13.5, <u>10.125</u>, 10.125, etc. The subsequence (underlined) of every other term beginning at 18, which is 18, 13.5, 10.125, etc., is an infinite geometric progression.

We can find its sum and double this value, then add 24. This is an infinite geometric series with $a_1 = 18$ and $r = 0.75$, so $S = \frac{18}{1 - 0.75} = 72$. Thus, the ball travels $24 + 2(72) = 168$ feet before coming to rest on the floor.

Mastery points

Can you

- Expand sigma expressions into series?
- Find the sum of finite arithmetic and geometric series?
- Find the sum of certain infinite geometric series?
- Solve applications using arithmetic and geometric series?

Exercise 12–2

Expand the following sigma expressions.

1. $\sum_{j=1}^{4}(4j + 1)$ **2.** $\sum_{j=3}^{5}(3j^2 + 1)$ **3.** $\sum_{j=2}^{5}j(j + 1)$ **4.** $\sum_{j=1}^{3}(j + 3)(j - 1)$ **5.** $\sum_{j=3}^{4}\frac{j}{j + 1}$

6. $\sum_{j=3}^{5}\frac{3j - 2}{j}$ **7.** $\sum_{j=1}^{6}(-1)^j\left(\frac{4}{3j}\right)$ **8.** $\sum_{j=1}^{4}(-1)^j(j + 1)^j$ $\boxed{9.}$ $\sum_{j=1}^{4}\left(\sum_{k=1}^{j}k^2\right)$ **10.** $\sum_{j=1}^{3}\left(\sum_{k=1}^{j}(k - 1)^2\right)$

Find the sum of the series determined by the given arithmetic sequence.

11. 3, 6, 9, . . . , 96
12. 2, 8, 14, . . . , 62
13. −10, −14, −18, . . . , −66
14. $\frac{1}{3}, \frac{2}{3}, 1, 1\frac{1}{3}, \ldots, 20\frac{2}{3}$
15. $-8, -7\frac{1}{4}, -6\frac{1}{2}, \ldots, 1$
16. −20, −5, 10, . . . , 85
17. $a_1 = 3, d = -5$; find S_{12}
18. $a_1 = -\frac{3}{4}, d = -\frac{1}{4}$; find S_{30}
19. $a_1 = -2, d = 2$; find S_{22}
20. $a_1 = 3\frac{1}{3}, d = -2\frac{2}{3}$; find S_{12}
21. 4, −2, −8, . . . ; find S_{14}
22. $-11, -9\frac{1}{2}, -8, \ldots$; find S_{21}
23. $a_1 = 2, a_3 = 7$; find S_{10}
24. $a_3 = 15, a_6 = 30$; find S_6
25. $a_5 = 50, a_8 = 68$; find S_6

Find the required *n*th partial sum for each geometric sequence.

26. $a_1 = 2, r = 3$; find S_4
27. $a_1 = -1, r = -2$; find S_{12}
28. $a_1 = \frac{2}{3}, r = \frac{1}{3}$; find S_7
29. $a_1 = -\frac{1}{4}, r = 1\frac{1}{4}$; find S_4
30. $-2, \frac{4}{3}, -\frac{8}{9}, \ldots$; find S_4
31. 3, 18, 108, . . . ; find S_3
32. −5, 15, −45, . . . ; find S_6
33. $\frac{8}{27}, \frac{4}{9}, \frac{2}{3}, \ldots$; find S_9
34. 3, 6, 12, . . . , 96
35. 2, 6, 18, . . . , 4,374
36. $\sum_{k=1}^{7} 3^k$
37. $\sum_{k=1}^{6} \frac{1}{9}(3^k)$
38. $\sum_{k=1}^{10} (-2)^k$
39. $\sum_{k=1}^{5} (\frac{2}{5})^k$
40. $\sum_{k=1}^{8} -3(-\frac{2}{3})^k$
41. $\sum_{k=1}^{12} \frac{16}{3}(\frac{1}{2})^k$

Find the sum of the given infinite geometric series. If the series has no sum state that.

42. $\sum_{i=1}^{\infty} (\frac{2}{3})^i$
43. $\sum_{i=1}^{\infty} 3(\frac{1}{3})^i$
44. $\sum_{i=1}^{\infty} -4(\frac{1}{2})^i$
45. $\sum_{i=1}^{\infty} \frac{2}{3}(\frac{1}{3})^i$
46. $\sum_{i=1}^{\infty} \frac{1}{8}(2^i)$
47. $\sum_{i=1}^{\infty} (-\frac{2}{3})^i$
48. $\sum_{i=1}^{\infty} (-\frac{9}{10})^i$
49. $\sum_{i=1}^{\infty} (\frac{4}{3})^i$
50. $14 + 7 + \frac{7}{2} + \cdots$
51. $3 + 2 + \frac{4}{3} + \cdots$
52. $4 + 5 + \frac{25}{4} + \cdots$
53. $1 - \frac{2}{3} + \frac{4}{9} - \cdots$

Find the rational number form of the repeating decimal number.

54. $0.666\overline{6}$
55. $0.222\overline{2}$
56. $0.5151\overline{51}$
57. $0.2828\overline{28}$
58. $0.216216\overline{216}$
59. $0.882882\overline{882}$
60. $0.21342134\overline{2134}$
61. $0.51555155\overline{5155}$
62. $0.23636\overline{36}$
63. $0.343535\overline{35}$
64. $0.216060\overline{60}$
65. $0.4014\overline{14}$

Solve the following problems.

66. A ball is dropped from a height of 10 meters. Each time it strikes the floor, the ball rebounds to a height that is 40% of the previous height. Find the total distance that the ball travels before it comes to rest on the floor.

67. A pendulum swings a distance of 20 inches on its first swing. Each subsequent swing is 95% of the previous distance. How far does the pendulum swing before it stops?

68. Neglecting air resistance, a freely falling body near the surface of the earth falls vertically 16 feet during the first second, 48 feet during the second second, 80 feet during the third second, and so on. Under these conditions how far will a body fall in the eighth second?

69. Referring to problem 68, how far will a body fall in 8 seconds?

70. A freely falling body in a vacuum near the surface of the earth will fall 4.9 m (meters) in the first second, 14.7 m in the second second, 24.5 m in the third second, and in general will fall 9.8 m farther in a given second than in the previous second. How far will such a body fall in 15 seconds?

71. How long will it take a freely falling body in a vacuum near the surface of the earth to fall 250 meters? Find the time to the nearest second. (See problem 70.)

72. An aircraft is flying at 400 mph, and is being followed at a distance of 1 mile (5,280 ft) by another aircraft moving at the same velocity. This second aircraft begins to accelerate so that each second it covers 30 more feet than it covered in the previous second. How long will it take before the second aircraft overtakes the first, from the time the second aircraft begins to accelerate?

73. A biologist in a laboratory estimates that a culture of bacteria is growing by 12% per hour. How long will it be, to the nearest hour, before the population doubles?

74. A certain bacterial culture triples in size each hour. If there were originally 1,000 bacteria how many hours would it take the bacteria to (a) surpass 1 million (b) surpass 10 million?

75. After doing a monarch a big favor one of the monarch's subjects asked to be rewarded in the following manner. Take a chessboard and put one grain of wheat on the first square, two on the second, four on the third, eight on the fourth, and so on, doubling the amount each time. The subject asked for the wheat that would be on the board. How many grains of wheat is this? (A chessboard has 64 squares.)

76. In a grocery store a clerk stacks boxes of cereal in a floor display so that there are 30 boxes of cereal in the first row, 27 in the second, 24 in the third, etc. How many boxes of cereal are in the floor display?

77. The same clerk as in problem 76 has 400 boxes to stack in a similar manner, except that each row is to have two fewer boxes than the one below it. The top row should have three boxes. How many boxes should be put in the first row to begin this display (the clerk may not be able to use all the boxes, but wants the display to be as high as possible)?

78. A parent put $500 into a bank account on the day a child was born. On each birthday the parent put in $100 more than the last deposit. How much money was deposited up to and including the child's eighteenth birthday?

79. It is estimated that over a 10-year period in a certain country the value of money went down 5% per year; that is, each year a dollar (the country's currency is in dollars) bought only 95% of what it bought the year before. By what percentage did the value of the dollar fall in this 10-year period?

80. A company needs $100,000 six years from now. It plans to obtain the money by making six deposits in an account. It will withdraw the interest earned each year, so this can be neglected. However, the company is growing and estimates that it can make each deposit 15% larger than the previous deposit. How large should the first deposit be?

81. A well-drilling company charges for well drilling according to the following schedule. One dollar for the first foot, and an additional $0.25 per foot for each foot after that. What would it charge to drill a well 250 feet deep?

82. A vacuum pump removes one-fifth of the remaining air in a tank with each stroke. **a.** How much air remains in the tank after the fifth stroke? **b.** How many strokes would be necessary to remove 98% of the air?

83. A donor to a college's development fund gave $50,000 and promised to give 80% of the previous year's donation each year, for an indefinite period. Excluding considerations such as inflation and the like, what is the value of this grant?

84. Derive the formula for the sum of a finite geometric series. Do this by writing the sum S as $S = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1}$. Then compute rS, and consider the sum $S - rS$.

85. Add up the integers from 1 to 100.

86. A basic formula in financial mathematics is the *present value formula*. Present value is the amount of money that would need to be invested at some rate of return to achieve some predetermined return in the future. If C_1 represents cash flow for period 1, C_2 for period 2, etc., and r_1 the rate of return (a percentage) that could be received on C_1, r_2 the rate of return that could be received on C_2, etc., then the present value, PV is

$$PV = \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2} + \frac{C_3}{(1 + r_3)^3} + \cdots$$

The *perpetuity formula* assumes that the cash flows and rates of return are all the same. If these values are C and r, then under these conditions

$$PV = \frac{C}{1 + r} + \frac{C}{(1 + r)^2} + \frac{C}{(1 + r)^3} + \cdots$$

Note that if $r > 0$, $1 + r > 1$, and therefore $\frac{1}{1 + r} < 1$.

Show that the present value in the perpetuity formula reduces to $\frac{C}{r}$.

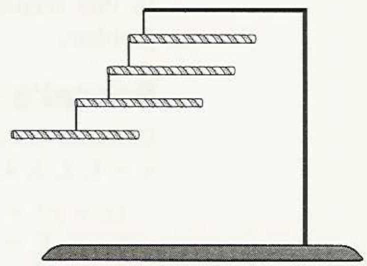
87. Two trains are 200 miles apart, headed toward each other on the same track, each traveling at 20 mph. A fly leaves one of the trains and flies directly toward the other at 60 mph (it is a very athletic fly). When it gets to the other train it turns around and repeats the journey to the first train. It repeats this process until the trains crash. How far did the fly fly?

88. On an ancient Babylonian tablet the value $1 + 2 + 2^2 + 2^3 + \cdots + 2^9$ is summed. Find this value.

89. How many record-breaking total yearly snowfalls (relative to their own life) would a person expect to see in a lifetime?[4] We will assume that the amount of snowfall in one year is unrelated to the amount that fell in the previous year. In the first year, the chance of a record is one out of one, or 1; the second year has a fifty-fifty chance of being more or less than the previous year (or one out of two, or $\frac{1}{2}$). In the third year the chance is $\frac{1}{3}$ (one out of three), since the heaviest snowfall for those 3 years could have fallen in any one of them. Similarly, in the nth year the chance of a new record is $\dfrac{1}{n}$. To find the total number of record snowfalls we must add up the sequence of probable record snowfalls for each year: $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \dfrac{1}{n}$. Find the expected number of record snowfalls that a person will see in their first 10 years of life.

[4]Problems 89 and 90 are extracted from "Snowfalls and Elephants, Pop Bottles, and π," by Ralph Boas, from *The Two-Year College Mathematics Journal,* Vol. 11, No. 2, March 1980. Boas attributes problem 90 to R. C. Buck, University of Wisconsin.

90. A mobile is being constructed of straws (see figure). It is desired to arrange things so that the bottom straw projects beyond the point of support. It turns out that, measured in straw-lengths, the successive offsets from the bottom should be $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$, etc. Since $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} \approx 1.042$, it takes just four straws to achieve this effect. How many straws would it take to achieve a $1\frac{1}{2}$ straw offset?



91. The following problem[5] comes from an ancient Hindu manuscript excavated in Pakistan in 1881. A certain person travels 5 yojanas (1 yojana = 8,000 times the height of a person) on the first day, and 3 yojanas more than the previous day on each successive day. Another person has a head start of 5 days, and travels 7 yojanas a day. In how many days will they meet?

92. A familiar nursery rhyme is:

As I was going to St. Ives,
I met a man with seven wives;
Every wife had seven sacks,
Every sack had seven cats,
Every cat had seven kits,
Kits, cats, sacks and wives,
How many were going to St. Ives?

[5]From "Hindu Romance with Quadratic Equations" by Dr. Gurcharan Singh Bhalla, *The Amatyc Review,* Fall 1987.

Skill and review

1. Find the 33rd term of an arithmetic series with $a_1 = 3$ and $d = 5$.
2. Find the fifth term of a geometric series with $a_1 = 3$ and $r = 5$.
3. Solve $\left|\dfrac{x+2}{x}\right| < 4$.
4. Graph $f(x) = x^2 + 5x - 6$.
5. Find the equation of the circle that has center at $(2, -1)$ and passes through the origin.
6. Graph $f(x) = x^3 - 3x^2 + x + 2$.

12–3 The binomial expansion and more on sigma notation

> When a certain computer program runs, the number of steps that it requires to process k data elements is given by $\sum_{i=1}^{k}(2i^2 + 5i - 12)$. Find an expression for this quantity in terms of k.

In this section we study the mathematics related to solving this type of problem.

Pascal's triangle

Consider the sequence of indicated products for the expression $(x + y)^n$, for $n = 1, 2, 3, 4, \ldots$.

$$\begin{aligned}
(x + y)^1 &= x + y \\
(x + y)^2 &= x^2 + 2xy + y^2 \\
(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x + y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \quad \text{etc.}
\end{aligned}$$

```
            1   1
          1   2   1
        1   3   3   1
      1   4   6   4   1
    1   5  10  10   5   1
            etc.
```

Figure 12–2

The coefficients can be found using **Pascal's triangle**[6] (figure 12–2). It can be seen that the next row in this tableau of numbers is formed by adding the elements of the row above it two at a time. For example, (as shown in figure 12–2) $1 + 1 = 2$, $1 + 3 = 4$, $6 + 4 = 10$.

To expand $(x + y)^k$ we might note the following.

- Row k of Pascal's triangle is the numeric coefficients.
- The exponent for x begins in the leftmost term with k, and the exponent for y is zero.
- As we move from one term to the following term the exponent for x decreases by one and that for y increases by one.
- The sum of the exponents in each term is always k.
- There are $n + 1$ terms.

Thus, we could compute $(x + y)^6$ by forming the next row of Pascal's triangle, which would be 1 6 15 20 15 6 1, and forming the terms

$$1x^6y^0 + 6x^5y^1 + 15x^4y^2 + \cdots \text{ etc.}$$

It would be difficult to determine, say, the third term of $(x + y)^{20}$ in this manner. We do know it would be of the form $Kx^{18}y^2$, but finding K would require determining the coefficients for $n = 1, 2, 3, \ldots, 19$ first. There is a formula which can be used to find these coefficients and expand $(x + y)^n$ at the same time. To understand it we need to define **factorials** first.

[6]Used by Blaise Pascal (1623–1662), a French mathematician. It is also depicted at the front of Chu Shih-Chieh's *Ssü-yüan yü-chien (Precious Mirror of the Four Elements),* which appeared in China in 1303.

Factorials

> ***n*-factorial**
> n-factorial, $n!$, is defined as
> $$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 \text{ for } n \geq 1, n \in N$$
> As a special case, $0! = 1$.

For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

and
$$\begin{aligned} 8! &= 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 8 \cdot 7 \cdot 6 \cdot 5! \\ &= 366 \cdot 120 = 40{,}320 \end{aligned}$$

Note Most calculators will calculate $n!$ with a key marked $x!$. On the TI-81 this is $\boxed{\text{MATH}}$ 5.

Combinations

Factorials are useful in what is called **combinatorics,** an area of mathematics and statistics that concerns itself with counting in complicated situations. For example, combinatorics is used to find the number of ways a person can win a lottery, or the number of trials a computer program may make to solve a given problem.

One thing that is important in combinatorics as well as our development here is called **combinations.** As an example, the number of ways to form a committee of 3 from a group of 8 people is called "the number of combinations of 3 things taken from 8 available things," or "8 choose 3," which, it turns out, is computed as

$$\frac{8!}{(8-3)! \cdot 3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

This computation is also written $\binom{8}{3}$ and is defined in general as follows.

> **Combinations**
> The number of k-element combinations of n available things, denoted by $\binom{n}{k}$ ("n choose k") is
> $$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$
> if $n \geq k \geq 0$, $k, n \in N$

It is useful to know that

$$[1]\quad \binom{n}{n} = 1 \qquad [2]\quad \binom{n}{1} = n \qquad [3]\quad \binom{n}{0} = 1$$

It is left as an exercise to verify that these statements are true.

Note Most calculators will calculate $\binom{n}{r}$ with a key marked $_nC_r$. On the TI-81 this is $\boxed{\text{MATH}}$ PRB 3.

■ ***Example 12–3 A***

Expand and simplify each expression.

1. $\binom{7}{3} = \frac{7!}{4!3!}$ Definition of $\binom{n}{k}$

$$= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 7 \cdot 5 = 35$$

Calculator: 7 $\boxed{_nC_r}$ 3 $\boxed{=}$

TI-81: 7 $\boxed{\text{MATH}}$ PRB 3 3 $\boxed{\text{ENTER}}$

2. $\binom{n+2}{n} = \frac{(n+2)!}{[(n+2)-n]! \cdot n!}$ Definition of $\binom{n}{k}$

$$= \frac{n! \cdot (n+1) \cdot (n+2)}{2! \cdot n!}$$

$$= \frac{(n+1)(n+2)}{2} = \frac{n^2 + 3n + 2}{2}$$

The binomial expansion formula

With this notation at hand[7] we can state the binomial expansion formula.

Binomial expansion formula

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i,\ n \in N$$

This formula incorporates the information about the coefficients of each term from Pascal's triangle and the observations we made about the exponents of each factor in a given term.[8] Its application is illustrated in example 12–3 B.

[7]The symbol $n!$ was introduced in 1808 by Christian Kramp of Strasbourg, in his *Élémens d' arithmétique universelle.* In 1846, Rev. Harvey Goodwin introduced the notation $\underline{n|}$ for the same thing. This notation may still be found in older books. Euler used the notation $\left(\frac{n}{k}\right)$ in 1778 and nineteenth-century writers shortened this to the current form $\binom{n}{k}$.

[8]A (difficult) proof of this formula uses the method of finite induction (section 12–4).

■ ***Example 12–3 B***

1. Expand $(2a - b)^4$ using the binomial expansion formula.

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i \quad \text{Binomial expansion formula}$$

$$(2a - b)^4 = \sum_{i=0}^{4} \binom{4}{i} (2a)^{4-i}(-b)^i \quad \text{Replace } x \text{ with } 2a,\ y \text{ with } -b, \text{ and } n \text{ with } 4$$

$$= \binom{4}{0}(2a)^{4-0}(-b)^0 + \binom{4}{1}(2a)^{4-1}(-b)^1 + \binom{4}{2}(2a)^{4-2}(-b)^2$$

$$+ \binom{4}{3}(2a)^{4-3}(-b)^3 + \binom{4}{4}(2a)^{4-4}(-b)^4$$

$$= 1(2a)^4(-b)^0 + 4(2a)^3(-b) + 6(2a)^2(-b)^2 + 4(2a)(-b)^3 + 1(2a)^0(-b)^4$$

$$= 16a^4 - 32a^3b + 24a^2b^2 - 8ab^3 + b^4$$

2. Find the fourth term in the expansion of $(x - 3y^2)^{10}$.
The fourth term results when $i = 3$, so the term needed is for $i = 3$, $n = 10$, and y replaced with $-3y^2$:

$$\binom{10}{3} x^{10-3}(-3y^2)^3 = -120x^7(27)y^6$$

$$= -3{,}240x^7y^6$$

■

Properties of sigma notation

Complicated uses of sigma notation are often encountered in computer science, economics, and statistics, as well as in advanced mathematics. In these cases, it is often useful to know some properties of sigma notation and to be able to manipulate the notation in certain ways.

> **Properties of sigma notation**
>
> Let k be a constant and let $f(i)$ and $g(i)$ represent expressions in the index variable i. Then the following properties are true:
>
> Sum of constants property: $\sum_{i=1}^{n} k = nk$
>
> Constant factor property: $\sum_{i=1}^{n} [k \cdot f(i)] = k \cdot \sum_{i=1}^{n} f(i)$
>
> Sum of terms property: $\sum_{i=1}^{n} [f(i) + g(i)] = \sum_{i=1}^{n} f(i) + \sum_{i=1}^{n} g(i)$

The sum of constants property states that if the expression is a constant, we get the product of the upper index and the constant. For example,

$$\sum_{i=1}^{4} 6 = 6 + 6 + 6 + 6 = 4(6) = 24$$

The constant factor property states that a common factor may be factored out of a sum. For example,

$$\sum_{i=1}^{3} 5i = 5 \cdot 1 + 5 \cdot 2 + 5 \cdot 3 = 5(1 + 2 + 3) = 5 \sum_{i=1}^{3} i$$

The sum of terms property states that the summation of an expression in two (or more, actually) terms is equivalent to summing each term separately. For example,

$$\begin{aligned}\sum_{i=1}^{3}(5i + i^2) &= (5\cdot 1 + 1^2) + (5\cdot 2 + 2^2) + (5\cdot 3 + 3^2)\\ &= (5\cdot 1 + 5\cdot 2 + 5\cdot 3) + (1^2 + 2^2 + 3^2)\\ &= \sum_{i=1}^{3} 5i + \sum_{i=1}^{3} i^2\end{aligned}$$

Sums of certain series

An expression for the sum of the series where the general term is i, i^2, and i^3 is known. The sum of integer series (below) is an arithmetic series, so its sum can be derived from the formula for the sum of an arithmetic series; this is assigned in the exercises. The expressions for the sum of squares of integers series and the sum of cubes of integers series (below) are proved in section 12–4.

Sums of certain series

Sum of integer series:	$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
Sum of squares of integers series:	$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
Sum of cubes of integers series:	$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

These sums, combined with the properties of sigma notation, can be used to sum certain series. This is shown in example 12–3 C.

■ ***Example 12–3 C***

Simplify the following series.

1. $\sum_{i=1}^{4}[5i + (\frac{4}{5})^i]$

$= \sum_{i=1}^{4} 5i + \sum_{i=1}^{4} (\frac{4}{5})^i$ Sum of terms property

$= 5\sum_{i=1}^{4} i + \sum_{i=1}^{4} (\frac{4}{5})^i$ Constant factor property

$\sum_{i=1}^{4} i = \frac{4\cdot 5}{2} = 10$ Sum of integer series

$\sum_{i=1}^{4} (\frac{4}{5})^i$ is a geometric series with $a_1 = r = \frac{4}{5} = 0.8$.

We want $S_4 = \dfrac{a_1(1 - r^n)}{1 - r} = \dfrac{0.8(1 - 0.8^4)}{1 - 0.8} = 2.3616.$

Thus, we continue:

$$= 5\sum_{i=1}^{4} i + \sum_{i=1}^{4} \left(\tfrac{4}{5}\right)^i = 5 \cdot 10 + 2.3616 = 52.3616$$

2. $\displaystyle\sum_{i=1}^{6} (2i - 3)^3$

$$= \sum_{i=1}^{6} (8i^3 - 36i^2 + 54i - 27)$$ Expand $(2i - 3)^3$

$$= 8\sum_{i=1}^{6} i^3 - 36\sum_{i=1}^{6} i^2 + 54\sum_{i=1}^{6} i - \sum_{i=1}^{6} 27$$ Sum of terms property and constant factor property

$$= 8\left[\frac{6(7)}{2}\right]^2 - 36\left[\frac{6(7)(13)}{6}\right] + 54\left[\frac{6(7)}{2}\right] - 6(27)$$ Sum of cubes, squares, and integer series, and sum of constants property

$$= 8(21^2) - 36(91) + 54(21) - 6(27) = 1{,}224$$

3. Find an expression for $\displaystyle\sum_{i=1}^{k} (i - 2)^2$, $k > 1$, in terms of the value k.

$$\sum_{i=1}^{k} (i - 2)^2 = \sum_{i=1}^{k} (i^2 - 4i + 4) = \sum_{i=1}^{k} i^2 - 4\sum_{i=1}^{k} i + \sum_{i=1}^{k} 4$$

$$= \frac{k(k+1)(2k+1)}{6} - 4\left(\frac{k(k+1)}{2}\right) + k \cdot 4$$

$$= \frac{2k^3 + 3k^2 + k}{6} - 2k^2 - 2k + 4k$$

$$= \tfrac{1}{3}k^3 + \tfrac{1}{2}k^2 + \tfrac{1}{6}k - 2k^2 - 2k + 4k$$

$$= \tfrac{1}{3}k^3 - \tfrac{3}{2}k^2 + \tfrac{13}{6}k$$

Mastery points

Can you

- State the definition of $\dbinom{n}{k}$?
- State the binomial expansion theorem?
- Use the three properties given for sigma notation to simplify appropriate expressions?
- Use the expressions given for $\sum_{i=1}^{n} i$, $\sum_{i=1}^{n} i^2$, and $\sum_{i=1}^{n} i^3$ to simplify appropriate series?

Exercise 12–3

Expand and simplify each expression.

1. $\dfrac{7!}{5!}$ **2.** $\dfrac{8!}{5!3!}$ **3.** $\dbinom{8}{5}$ **4.** $\dbinom{10}{3}$

5. $\dbinom{6}{6}$ **6.** $\dbinom{20}{16}$ **7.** $\dbinom{n+3}{n}$ **8.** $\dbinom{k}{k-1}$

Expand and simplify the following expressions using the binomial expansion formula.

9. $(ab - 3)^4$ **10.** $(ab^2 + 2c^3)^5$ **11.** $(2p^4 + q)^6$ **12.** $(a^4 + 2b^3)^5$

13. $(a^3b^2 - 2c)^7$ **14.** $(2p^2 - q)^6$ **15.** $\left(\dfrac{p}{2} + 2\right)^6$ **16.** $\left(3a - \dfrac{2b}{3}\right)^5$

17. Find the fifth term of $(a^3 + 2b^5)^{15}$.

18. Find the fourth term of $(p^4 - q)^{18}$.

19. Find the fourth term of $(p - 3q)^{22}$.

20. Find the third term of $\left(2a - \dfrac{b}{8}\right)^6$.

Compute the sum of the following series.

21. $\sum_{i=1}^{56} 8$ **22.** $\sum_{i=1}^{18} (i + 3)$ **23.** $\sum_{i=1}^{23} (4i - 6)$ **24.** $\sum_{i=1}^{9} (2i^2 - 4)$

25. $\sum_{i=1}^{9} (3 - 4i + i^2)$ **26.** $\sum_{i=1}^{9} (i - 4)^2$ **27.** $\sum_{i=1}^{12} (2i + 1)^2$ **28.** $\sum_{i=1}^{10} (i^3 - 4i + 2)$

29. $\sum_{i=1}^{7} (2i^3 - 3)$ **30.** $\sum_{i=1}^{9} (i - 3)^3$ **31.** $\sum_{i=1}^{8} (i^3 + 6i^2 + 8i - 1)$ **32.** $\sum_{i=1}^{8} (8i^3 - 16i^2)$

33. $\sum_{i=1}^{4} [i^2 - (\frac{1}{4})^i]$ **34.** $\sum_{i=1}^{6} [6i^2 + 3i - (\frac{1}{3})^i]$ **35.** $\sum_{i=1}^{6} [4(\frac{1}{4})^i - 2(\frac{3}{2})^i]$ **36.** $\sum_{i=1}^{5} [16(\frac{1}{2})^i - 3(\frac{3}{4})^i]$

37. Find an expression for $\sum_{i=1}^{k} (6i^2 - 4i + 2)$ in terms of k.

38. Find an expression for $\sum_{i=1}^{k} (12i^2 + 2i - 7)$ in terms of k.

39. When a certain computer program runs, the number of steps that it requires to process k data elements is given by $\sum_{i=1}^{k} (2i^2 + 5i - 12)$. Find an expression for this quantity in terms of k.

40. Find an expression for $\sum_{i=1}^{k} (i^2 - i - 1)$ in terms of k.

41. Create Pascal's triangle down to the eighth row. The first row is "1 1".

42. Add up the values in a few of the rows of Pascal's triangle. Make a conjecture about what the value is in the kth row.

43. Show that $\dbinom{n}{n} = 1$, $\dbinom{n}{1} = n$, and $\dbinom{n}{0} = 1$.

44. State the binomial expansion theorem.

45. Prove that $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$. Note that $1 + 2 + \cdots + n$ is an arithmetic series (section 12–2).

46. Show that $\dbinom{n}{k+1} + \dbinom{n}{k} = \dbinom{n+1}{k+1}$ for $n \ge k \ge 0$.

47. Show that $\dbinom{n}{k} = \dbinom{n}{n-k}$ for $n \ge k \ge 0$.

48. Show that $(1 + k)^n \ge 1 + nk$ for all values of $k \ge 0$ and for all integers $n \ge 0$. (*Hint:* Expand $(1 + k)^n$.)

49. Prove that $\sum_{i=0}^{n} \dbinom{n}{i} = 2^n$. (*Hint:* This is a form of the binomial expansion with $x = y = 1$.)

Skill and review

1. If $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$, what is $1 + 2 + \cdots + n + (n+1)$ equal to?
2. If $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$, what is $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} + \dfrac{1}{(n+1)(n+2)}$ equal to?
3. Find an expression for $\dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \cdots + \dfrac{1}{3^n}$.
4. Add up the even integers $2 + 4 + \cdots + 240$.
5. Solve $\dfrac{x-1}{2} - \dfrac{x-1}{3} = x$.
6. Graph $f(x) = (x-1)^3 - 1$.

12–4 Finite Induction

Prove that the sum of the squares of the first n natural numbers is $\dfrac{n(n+1)(2n+1)}{6}$. That is, prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

This series is not arithmetic or geometric, so the methods of the previous sections are ineffective. The formula was correctly guessed over 700 years ago, probably by trial and error. In this section we show a method that allows us to prove that the formula really does work for any value of n.

The series $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$ is an arithmetic series, and it can be shown that its sum is $\dfrac{n(n+1)}{2}$, using the methods of section 12–2 for summing arithmetic series. That is,

$$[1] \qquad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for any positive integer n.

We will prove this result here using the method of proof mentioned above, called **finite induction.** This is used to show that statements such as equation [1] are true for every positive integer. These types of statements arise often in advanced mathematics and the analysis of algorithms in computer science.

The idea of finite induction is as follows. Suppose we know that some statement is true for the integers 1, 2, and 3. For example:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \text{ is true for } n = 1, 2, 3.$$

Calculation will verify that this statement is true. For $n = 3$ the check is $1 + 2 + 3 = \dfrac{3(3+1)}{2}$, so $6 = 6$.

Now, suppose that we could prove the following: Whenever the statement above is true for an integer it also works for the next integer. This would mean that the statement must be true for $n = 4$, since we can see that it works for $n = 3$. Now, if it is true for $n = 4$, then the same logic says it must work for $n = 5$. We can proceed along these lines to any value of n we wish. This is the concept of finite induction.

Let us see if we can show that the supposition we used above is true. That is, can we show that, if [1] is true up to some integer k (statement [2]),

$$[2] \qquad 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

then it must also be true for the next integer, $k + 1$? The formula for $k + 1$ would be

$$[3] \qquad 1 + 2 + 3 + \cdots + k + (k+1) \stackrel{?}{=} \frac{(k+1)(k+2)}{2},$$

which we obtain by substituting $k + 1$ into equation [1] instead of k. *What we want to do is show that equation [3] is true, given that equation [2] is true.* (We show that a statement needs to be shown true by the symbol $\stackrel{?}{=}$.)

We proceed as follows. We know that equation [2] is true up to some value of k (in this case, 3). Now, add the next value, $k + 1$, to both members of equation [2]:

$$[4] \qquad 1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1).$$

Equation [4] must be true, since we have simply added the same quantity to both members of a true equation. Now, the left member of equation [4] is the same as the left member of equation [3]; if we could show that the right member of equation [4] is the same as the right member of equation [3], we would have shown that equation [4] is really equation [3], and, since equation [4] is true, so is equation [3]. Proceeding with the right member of equation [4],

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

which is the right member of equation [3].

Although this finishes our proof, it is unlikely that a first-time reader would understand completely what we have done, so let us review what happened.

1. We wanted to prove that the formula [1] is true *for any positive integer n.* Imagine picking a value for n.
2. We checked that equation [1] is true for $n = 1$, 2, and 3, by hand. We could check for even more values if we wished.

3. We thus knew that equation [1] was true up to at least a value of 3. Equation [2] is just equation [1] rewritten for $n = k$. Thus, equation [2] was true if $k = 3$ (and $k = 1$ or $k = 2$).
4. We showed that, if equation [2] was true for some arbitrary but fixed k, then so was equation [3].
5. Equation [2] is true for $k = 1, 2,$ or 3. Equation [3] states that, if equation [2] is true for 3, it is true for 4 (i.e., $k + 1$). Thus, equation [2] is true for 4.
6. Now, we know that equation [2] is also true for 4, and so equation [3] shows that equation [2] is true for 5 ($4 + 1$).
7. Repeating the logic of step 6, we can see that equation [2] is true for $k = 6, 7, 8$, etc. In fact, we can clearly repeat the steps to arrive at the conclusion that equation [2] is true for $k = n$, no matter how large n is. When $k = n$, equation [2] becomes equation [1], which is therefore true for n.

Observe that we really did not need to check equation [1] for n up to 3 by hand; just checking it for 1 would have been enough, since our logic would have then shown it must be true for 2 ($1 + 1$) and 3 ($2 + 1$).

With this example in mind, we state the principle of finite induction.

Principle of finite induction

If

1. a statement is true for $n = 1$ and
2. it can be shown that if the statement is true for $n = k$ then it must also be true for $n = k + 1$,

then the statement is true for any positive integer.

Example 12–4 A illustrates proofs by finite induction for situations in which we need to show that a certain sum of terms is equal to some expression.

■ ***Example 12–4 A***

1. Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

First, show the statement is true for $n = 1$:

$$1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}, \text{ so } 1 = 1$$

Next, assume the statement is true for $n = k$ ($k = 1$, for example), and then show that this implies that the statement must be true for $n = k + 1$. Assume that

$$[1]\ 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

is true, and use this to prove that the same statement is true for $k + 1$; for $k + 1$ this is

$$[2] \quad 1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 \stackrel{?}{=} \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

or

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 \stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6}$$

Let us call this statement (equation [2]) our **"goal statement."**

Proceed as follows. Assume equation [1] is true for k. Now, add the next term, $(k+1)^2$, to both members of equation [1].

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

The left member is the same as the left member of the goal statement, equation [2]. We need to show that the right member is the same as the right member of the goal statement.

$$\begin{aligned}\frac{k(k+1)(2k+1)}{6} + (k+1)^2 &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}\\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}\\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6}\\ &= \frac{(k+1)(k+2)(2k+3)}{6}\end{aligned}$$

which is the right member of the goal statement.

Thus, we have shown that the original statement is true for $n = 1$, and that, if it is true for $n = k$, it must also be true for $n = k + 1$, thus proving, by the principle of finite induction, that the statement is true for any value of n.

2. Prove by induction that

$$\frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \frac{1}{12 \cdot 17} + \cdots + \frac{1}{(5n-3)(5n+2)} = \frac{n}{2(5n+2)}$$

First the case where $n = 1$:

$$\tfrac{1}{14} = \tfrac{1}{14}$$

Now assume the statement is true for $n = k$:

$$[1] \quad \frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \frac{1}{12 \cdot 17} + \cdots + \frac{1}{(5k-3)(5k+2)} = \frac{k}{2(5k+2)}$$

We want to show the statement is true for $n = k + 1$; the goal statement is

$$[2] \quad \frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \frac{1}{12 \cdot 17} + \cdots + \frac{1}{(5(k+1)-3)(5(k+1)+2)} \stackrel{?}{=} \frac{k+1}{2(5(k+1)+2)} \text{ or}$$

$$\frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \frac{1}{12 \cdot 17} + \cdots + \frac{1}{(5k+2)(5k+7)} \stackrel{?}{=} \frac{k+1}{2(5k+7)}$$

Add the next term to both members of equation [1].

$$\frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \frac{1}{12 \cdot 17} + \cdots + \frac{1}{(5k-3)(5k+2)} + \frac{1}{(5(k+1)-3)(5(k+1)+2)}$$
$$= \frac{k}{2(5k+2)} + \frac{1}{(5(k+1)-3)(5(k+1)+2)} \text{ or}$$

$$\frac{1}{2 \cdot 7} + \frac{1}{7 \cdot 12} + \frac{1}{12 \cdot 17} + \cdots + \frac{1}{(5k+2)(5k+7)} = \frac{k}{2(5k+2)} + \frac{1}{(5k+2)(5k+7)}.$$

The left member is the same as the left member of the goal statement [2], so if the right members are the same, then equation [2] is true. We simplify the right member:

$$\frac{k}{2(5k+2)} + \frac{1}{(5k+2)(5k+7)} = \frac{k(5k+7)}{2(5k+2)(5k+7)} + \frac{1(2)}{2(5k+2)(5k+7)}$$
$$= \frac{5k^2+7k+2}{2(5k+2)(5k+7)} = \frac{(k+1)(5k+2)}{2(5k+2)(5k+7)}$$
$$= \frac{k+1}{2(5k+7)}$$

This is the right member of the goal statement, equation [2], so equation [2] is correct whenever equation [1] is correct. Thus, the original statement is true. ■

The principle of finite induction can be applied to situations in which we are not simply showing that some summation formula is true. These applications are harder to understand, and may require some facts beyond the algebraic manipulations shown above. For example we will show that $n^2 + n$ is always divisible by 2, when n is a positive integer, in example 12–4 B.

We will use the fact that divisibility by a certain value means that that value is a factor. Thus, we will use statements like:

If an integer j is divisible by 2, then $j = 2m$ for some integer m.

If an integer j is divisible by 3, then $j = 3m$ for some integer m.

We may also find it easier to work from the goal statement.

Example 12–4 B Prove by induction that $n^2 + n$ is divisible by 2 for all $n \varepsilon N$.

First, show that the statement is true when $n = 1$: $1^2 + 1 = 2$ which is divisible by 2. Now, assume that the statement is true for $n = k$; that is, assume that $k^2 + k$ is divisible by 2. Our goal is to show that this implies that $(k + 1)^2 + (k + 1)$ is divisible by 2. In this case, it is more convenient to work from the goal statement.

$$\begin{aligned}(k + 1)^2 + (k + 1) &= k^2 + 3k + 2\\ &= (k^2 + k) + (2k + 2)\end{aligned}$$

Write this last expression this way because we have the expression $k^2 + k$ in mind.

Now, since $k^2 + k$ is divisible by 2, it can be written as $2m$ for some integer m; thus we can proceed:

$$\begin{aligned}&= 2m + 2k + 2 &&\text{Replace } k^2 + k \text{ by } 2m\\ &= 2(m + k + 1)\end{aligned}$$

which is clearly divisible by 2.

Thus, assuming that $n^2 + n$ is divisible by 2 for some k, we have shown that it is also divisible by 2 for the next integer, $k + 1$. Hence by induction $n^2 + n$ is divisible by 2 for all $n \varepsilon N$. ■

Example 12–4 B shows that induction can be used for statements other than simple formulas.

A note to the skeptic[9]

After seeing this method of proof for the first time it might seem that almost anything, whether true or not, can be proved this way. This is not the case, however. Two examples will illustrate.

First, consider the statement $1 + 3 + 5 + \cdots + (2n - 1) = \frac{n^2 + n}{2}$ for every positive integer. This statement can be shown true for $n = 1$, but if we assume it is true for k we will not be able to. This is because the original statement is false! Now consider the statement $4 + 10 + 16 + \cdots + (6n - 2) = 3n^2 + n - 2$, which is not even true for $n = 1$, but, if assumed true for k can be shown true for $k + 1$! The reader is invited to explore both of these examples in the exercises.

These two examples do not "prove" that proof by induction works only on true statements, but hopefully will convince the reader that this is in fact the case. The principle of finite induction can be proved true in higher mathematics.

[9] "There is no use trying," (Alice) said. "One can't believe impossible things." "I daresay you haven't had much practice," said the Queen. "When I was your age, I always did it for half-an-hour a day. Why, sometimes I've believed as many as six impossible things before breakfast." (From *Alice in Wonderland,* by Lewis Carroll.)

Mastery points

Can you

- Prove statements using finite induction?

Exercise 12–4

Prove that the following statements are true for all $n \in N$ using finite induction.

1. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$

2. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

3. $4 + 9 + 14 + \cdots + (5n - 1) = \dfrac{n(5n + 3)}{2}$

4. $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$

5. $1 + 5 + 9 + \cdots + (4n - 3) = 2n^2 - n$

6. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$ (See footnote 10.)

7. $1 + 8 + 30 + 80 + \cdots + \dfrac{n^2(n + 1)(n + 2)}{6} = \dfrac{n(n + 1)(n + 2)(n + 3)(4n + 1)}{120}$

8. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$

9. $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n} = \dfrac{2^n - 1}{2^n}$

10. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n + 1)}{2}\right]^2$

11. Show that $n^3 + 2n$ is divisible by 3 for any natural number n.

12. Show that $(1 + a)^n \geq 1 + na$ for any natural number n, assuming $a \geq 0$.

13. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$

14. $1 + 4 + 4^2 + \cdots + 4^{n-1} = \dfrac{4^{n-1}}{3}$

15. $2 + 6 + 18 + \cdots + 2(3^{n-1}) = 3^n - 1$

16. $3 + 12 + 48 + \cdots + 3(4^{n-1}) = 4^n - 1$

17. $8 + 4 + 2 + \cdots + \dfrac{1}{2^{n-4}} = \dfrac{2^n - 1}{2^{n-4}}$

18. $\dfrac{1}{2 \cdot 5} + \dfrac{1}{5 \cdot 8} + \dfrac{1}{8 \cdot 11} + \cdots + \dfrac{1}{(3n - 1)(3n + 2)} = \dfrac{n}{6n + 4}$

19. $\dfrac{1}{1 \cdot 4} + \dfrac{1}{4 \cdot 7} + \dfrac{1}{7 \cdot 10} + \cdots + \dfrac{1}{(3n - 2)(3n + 1)} = \dfrac{n}{3n + 1}$

20. $\dfrac{1}{a(a + b)} + \dfrac{1}{(a + b)(a + 2b)} + \dfrac{1}{(a + 2b)(a + 3b)} + \cdots + \dfrac{1}{[a + (n - 1)b](a + nb)} = \dfrac{n}{a(a + nb)}$

21. $\dfrac{1}{1 \cdot 2 \cdot 3} + \dfrac{1}{2 \cdot 3 \cdot 4} + \dfrac{1}{3 \cdot 4 \cdot 5} + \cdots + \dfrac{1}{n(n + 1)(n + 2)} = \dfrac{n(n + 3)}{4(n + 1)(n + 2)}$

22. $\dfrac{1}{1 \cdot 3 \cdot 5} + \dfrac{1}{3 \cdot 5 \cdot 7} + \dfrac{1}{5 \cdot 7 \cdot 9} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)(2n + 3)} = \dfrac{n(n + 2)}{3(2n + 1)(2n + 3)}$

[10]Carl Boyer notes that the Babylonians may have known this result thousands of years ago. Also, this and the next problem are both found in the *Precious Mirror of the four Elements*, a book that appeared in China in 1303.

23. $\dfrac{1}{1 \cdot 4 \cdot 7} + \dfrac{1}{4 \cdot 7 \cdot 10} + \dfrac{1}{7 \cdot 10 \cdot 13} + \cdots + \dfrac{1}{(3n-2)(3n+1)(3n+4)} = \dfrac{n(3n+5)}{8(3n+1)(3n+4)}$

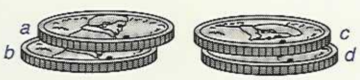
24. $(1 \cdot 2) + (2 \cdot 3) + \cdots + [n(n+1)] = \dfrac{(n+1)^3 - (n+1)}{3}$

25. In the text we considered the statement $1 + 3 + 5 + \cdots + (2n-1) = \dfrac{n^2+n}{2}$ where we said that this statement can be shown true for $n = 1$, but if we assume it is true for k and try to show it true for $k+1$ we will not be able to. Try this and see what happens.

26. Also in the text we stated that the statement $4 + 10 + 16 + \cdots + (6n-2) = 3n^2 + n - 2$ is not true even for $n = 1$, but, if assumed true for k can be shown true for $k+1$. Show that this is indeed the case.

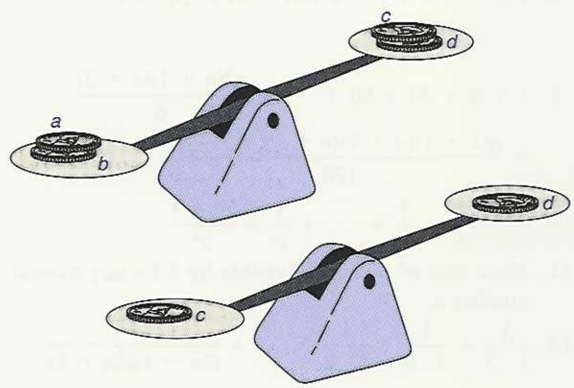
27. Refer to problems 25 and 26. The two series (a) $1 + 3 + 5 + \cdots + (2n-1)$ and (b) $4 + 10 + 16 + \cdots + (6n-2)$ are arithmetic series. Find an expression for the sum of the first n terms of each series. See section 12–2 for arithmetic series, if necessary. (Note that we are "fixing" the previous two problems by finding the correct expression for the right member of each equation.)

$\boxed{28.}$ You are given n coins ($n \geq 2$). They look identical, but one of them is counterfeit and weighs less than all of the others. You have a balancing scale, and need to determine which of the coins is counterfeit.



If you have two coins you can detect the light coin in one weighing. Three coins can be done in one weighing: weigh two coins; if they balance, the third coin is the counterfeit, otherwise the balance shows the light counterfeit coin. Four coins would require two weighings as shown in the diagram. The first weighing shows that the light coin is c or d. The second weighing shows that d is the light coin.

Two weighings would suffice for five coins also. We can show this inductively, since we can group the coins into a group of four coins and one coin. If the light coin is in the group of four, two weighings will be enough. If the four are the same, the fifth coin is the light coin.



In this manner we can show inductively that two weighings suffice for any number of coins. For n coins, group them into a group of $n-1$ coins and 1 coin. If the lighter coin is not among the $n-1$ coins, which takes two or fewer weighings, then it is the single coin.

Unfortunately, this statement is not true. Try to actually apply this idea to six coins to see that it is false. Then explain where the logic we applied was in error.

Skill and review

1. If $1 + 4 + 7 + \cdots + (3n-2) = \dfrac{n(3n-1)}{2}$, what is $1 + 4 + 7 + \cdots + (3n-2) + [3(n+1) - 2]$?
2. Graph $3x - 4y = 12$.
3. Graph $3x^2 - 4y^2 = 12$.
4. Solve $\begin{matrix} 2x - 3y \leq 12 \\ x + 2y \geq 4 \end{matrix}$.
5. Solve $2x - 1 > \dfrac{5}{x+1}$.

12–5 Introduction to combinatorics

Suppose NASA has 19 astronauts suitable for the next space mission. A crew consists of five astronauts. How many different crews are possible?

The answer to questions like this are found using counting methods that are introduced in this section. These methods are important in the study of probability, which is studied in section 12–6.

The basic concepts of probability depend on our ability to determine the number of possible ways that an experiment (such as rolling a die eight times) can occur. We must know what is possible before we can determine what is probable.

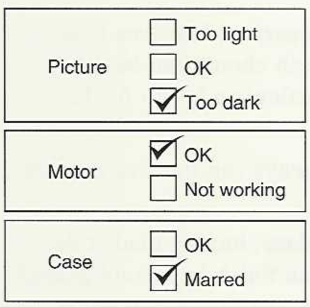


Figure 12–3

Multiplication of choices

To illustrate one of the basic principles of counting, consider an assembly line that produces a camera. A final test is made of each camera. A picture is taken, and it might be too light, alright, or too dark. The motor may or may not work, and the case may be marred or not. Each of these characteristics is marked on a slip. In how many ways can this slip be marked? Figure 12–3 shows one such marking.

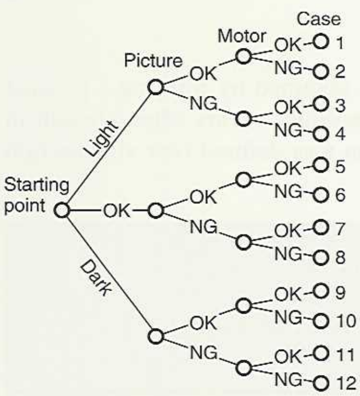


Figure 12–4

The possible markings can be displayed on a tree diagram (figure 12–4); NG stands for no good in the figure (anything that is not ''OK''). From a starting point we represent the three qualities of a picture, then for each of these possibilities the two characteristics of a motor, and for each of these (now 6) possibilities the two conditions of the case. This gives 12 possible markings of each slip.

These markings can also be shown with a list. If we list the picture quality first, then the motor, then the case, using L = light, O = OK, D = dark, N = no good, we obtain the list *LOO, LON, LNO, LNN, OOO, OON, ONO, ONN, DOO, DON, DNO, DNN.* If we had chosen to list the markings for, say, the motor first this would still be considered the same list, just as the tree diagram could have started with the first branch indicating the markings for the motor or case.

It is no coincidence that $12 = 3 \cdot 2 \cdot 2$. This example illustrates the following property, called the multiplication-of-choices property.

Multiplication-of-choices property
If a choice consists of k decisions, where the first can be made n_1 ways and for each of these choices the second can be made in n_2 ways, and in general the ith choice can be made in n_i ways, then the complete choice can be made in $n_1 \cdot n_2 \cdot \cdots \cdot n_k$ ways.

Each complete choice is called an **outcome.** Example 12–5 A illustrates using the multiplication-of-choices property.

Example 12–5 A

Determine the number of outcomes.

1. This year the Fine Arts Auditorium is hosting four plays and 12 concerts. A student pass entitles a student to see one play and one concert. In how many ways can a student see one play and one concert?

   Using the multiplication-of-choices property we calculate $4 \cdot 12$ or 48 different ways.

2. A newsstand carries five different newspapers, four different sporting magazines, six fashion magazines, and 12 general interest magazines. In how many ways can a shopper buy one of each of these types of magazines?

   In this case the choice of one of each of the magazines involves four choices. Using the number of ways in which each choice can be made, and the multiplication-of-choices property we calculate $5 \cdot 4 \cdot 6 \cdot 12 = 1{,}440$ ways.

3. A race has five individuals in it. In how many ways can the five runners finish the race (neglecting ties)?

   Any one of the five runners can finish in first place; having made this "choice," any of the four remaining runners can finish in second place. There are then three possibilities for third place, two for fourth place, and, finally, one choice for fifth place. Thus, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ different ways for the race to end. ■

Factorial notation

In part 3 of example 12–5 A the answer was obtained by forming a product of all the integers from 1 to 5; this type of product occurs often enough in counting problems that the following notation was defined (see also section 12–3).[11]

n-factorial

n-factorial, denoted by $n!$, is defined as

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 \text{ for } n \geq 1,\ n \in N$$

As a special case, $0! = 1$.

Thus, for example,

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 \qquad \text{Six-factorial is 720}$$
$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 720 = 3{,}628{,}800$$
$$1! = 1$$

We define $0!$ to be 1 strictly for convenience in certain formulas that we will see later.

[11]The symbol $n!$ was introduced by Christian Kramp of Strasbourg in 1808.

Observe that 6! is also $6 \cdot 5!$, or $6 \cdot 5 \cdot 4!$, or $6 \cdot 5 \cdot 4 \cdot 3!$, etc. In general

$$\begin{aligned} n! &= n(n-1)! \\ &= n(n-1)(n-2)! \\ &= n(n-1)(n-2)(n-3)!, \text{ etc.} \end{aligned}$$

We can make use of this fact to simplify certain expressions involving factorials, as illustrated in example 12–5 B. We also note that most calculators can calculate $n!$.

■ ***Example 12–5 B***

Simplify the following expressions.

1. $\dfrac{12!}{8!}$

$$\frac{12!}{8!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8!}{8!} = 12 \cdot 11 \cdot 10 \cdot 9 = 11{,}880$$

2. $\dfrac{15!}{12!3!}$

$$\frac{15!}{12!3!} = \frac{15 \cdot 14 \cdot 13 \cdot 12!}{12! \cdot 3 \cdot 2 \cdot 1} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2} = 5 \cdot 7 \cdot 13 = 455$$ ■

Permutations and combinations

A club with four members wants to choose a president and a secretary. In how many ways can this be done? Using the multiplication-of-choices property, we see that we have four choices to fill the president's position, and then there remain three choices to fill the secretary's position. Thus, there are $4 \cdot 3 = 12$ ways to fill these two positions. If the persons are person A, B, C, and D, we can list the 12 possible ways:

Selection	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
President	A	A	A	B	B	B	C	C	C	D	D	D
Secretary	B	C	D	A	C	D	A	B	D	A	B	C

Note that *the order of selection was important.* For example, the selection of A then B (selection 1) means that A is president and B is secretary, whereas the selection of B first then A (selection 4) means that B is president and A is secretary.

Suppose the same club wants instead to form a two-person committee. The same analysis shows all the ways in which we can select a two-person committee; however, we now don't care about who is selected first and who is selected second. Thus, selection 1 and selection 4 are equivalent as a two person committee. In fact, we can verify that in this case there are only half as many possible selections, or six different two-person committees. This is because we do not care about the $2 \cdot 1$ different ways each committee could be ordered.

In both of these situations a person could not be selected twice. This is called **selection without repetition (or without replacement).** This is the situation we will continue to consider here. In the situation of selecting a president and secretary, where *order is important,* each different selection is called a **permutation.**[12] In selecting committees, where *order is not important,* each selection is called a **combination.**

In selecting our president and secretary we would say we wanted the number of permutations of two things taken from four available things. This is written symbolically as ${}_4P_2$. Similarly in selecting our committees we wanted ${}_4C_2$, or the number of combinations of two things taken from four available things. We will develop formulas for each of these situations.

Permutations

Consider the following examples of ${}_nP_r$ for different values of n and r:

${}_5P_3 = 5 \cdot 4 \cdot 3$ Out of five people, choose a president, a vice-president and a secretary

${}_7P_4 = 7 \cdot 6 \cdot 5 \cdot 4$ Out of seven books choose four and arrange them on a shelf

We define ${}_nP_r$ in the following way.

Permutations[13]

The number of permutations of r distinct elements selected from n available elements, where $r \leq n$ is

$$ {}_nP_r = \underbrace{n(n-1)(n-2)\ldots(n-[r-1])}_{r \text{ factors}} $$

Note that this is the first r factors in $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$. Also, note that ${}_nP_n = n!$.

Note Most modern calculators have keys which will compute ${}_nP_r$.

In the exercises we illustrate by example that an alternative formula for ${}_nP_r$ is

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

This can be useful under certain circumstances, as illustrated in part 2 of example 12–5 C.

[12]The first known enumerations of permutations are from the Hebrew Book of Creation (c. A.D. 100), according to D. E. Knuth, *The Art of Computer Programming,* Vol. 3, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1973.

[13]The notation ${}_nP_r$ was introduced by Rev. Harvey Goodwin of Cambridge University circa 1869.

■ ***Example 12–5 C*** Determine the number of outcomes.

1. During periods of radio silence between two ships messages are sent by means of signal flags. If there are eight different flags, how many messages can be sent by placing three flags, one above the other, on a flagpole?

Since there is a definite order implied by the placement of the flags the problem involves permutations. From eight available things, choose three in order. The number of ways to do this is ${}_8P_3 = 8 \cdot 7 \cdot 6 = 336$ different messages.

Typical keystrokes on a calculator are 8 $\boxed{{}_nP_r}$ 3 $\boxed{=}$. On a TI-81 this function is under the $\boxed{\text{MATH}}$ key, in the PRB menu. Select 8 $\boxed{\text{MATH}}$ PRB 2 3 $\boxed{\text{ENTER}}$.

2. A trucking firm has 20 trucks and 18 drivers. The trucks differ in age and so are considered different by the drivers. In how many ways can the trucks be assigned to the drivers?

From 20 available trucks we need to select 18, and the order matters. Thus we need to compute ${}_{20}P_{18} = 20 \cdot 19 \cdot 18 \cdot \cdots \cdot 3$.

An $\boxed{{}_nP_r}$ calculator key will perform the calculation for us easily, but if a calculator does not have this key, but does have a factorial key $\boxed{x!}$ we could instead use the alternative form (from above):

$$
\begin{aligned}
{}_{20}P_{18} &= \frac{20!}{(20-18)!} = \frac{20!}{2!} = 1.216451004 \times 10^{18} \\
&\approx 1{,}216{,}451{,}004{,}000{,}000{,}000
\end{aligned}
$$

The result is so large we can only expect an approximate result using a calculator. The actual result, found with a sophisticated computer program,[14] is 1,216,451,004,088,320,000. ■

Indistinguishable permutations of *n* elements

Consider the possible arrangements of the letters of the word NOON. Since there are four letters we might conclude that there are ${}_4P_4 = 4! = 24$ ways to arrange the letters. However, the two O's and the two N's are indistinguishable. For example, the permutation in which the two N's are switched produces NOON also, which is indistinguishable from the original permutation.

[14]In this case Theorist®. Other powerful products that would calculate this number are Mathematica® and Maple.

For each arrangement, if the N's are only permuted among themselves, there is no distinguishable change; since there are 2! permutations of the two N's, we must divide the 24 by 2!. We must also divide by a second 2! for the O's. Thus, the number of distinguishable permutations of the letters in the word NOON is $\frac{4!}{2!2!} = 6$. We now make the following generalization.

> **Indistinguishable permutations**
>
> The number of distinct permutations P of n elements, where n_1 are alike of one kind, n_2 are alike of one kind, . . . , and n_k are alike of one kind, where $n_1 + n_2 + \cdots + n_k = n$, is given by
>
> $$P = \frac{n!}{n_1!n_2! \ldots n_k!}$$

Note that many of the n_i's may be one, which we can ignore in the computation. Example 12–5 D illustrates this idea.

■ ***Example 12–5 D***

Determine the number of outcomes.

1. Find the number of distinct permutations of the letters in the word MISSISSIPPI.

There are eleven letters of which there is one M, four I's, four S's, and two P's. Therefore the number of permutations is $P = \frac{11!}{1!4!4!2!} = 34{,}650$.

2. If a signal consists of nine flags one above the other on a flagpole, and there are nine flags, with three red, three white, and three blue, how many signals can be created?

$$P = \frac{9!}{3!3!3!} = 1{,}680$$

Combinations

Recall from the discussion of two-person committees that order is not important in combinations. A combination is the same as a subset of some set of distinguishable elements. As illustrated with committees, the number of combinations of n things taken r at a time ${}_nC_r$ is the same as ${}_nP_r$ if we divide by the number of permutations of r things. (For the committees, r was 2.) Thus, ${}_nC_r = \frac{{}_nP_r}{r!}$. It can be shown that this is the same value as $\frac{n!}{r!(n-r)!}$.

Combinations

The number of combinations (subsets) of r distinct elements selected from n available elements, where $r \leq n$ is

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

Another notation[15] for ${}_nC_r$ is $\binom{n}{r}$.

Note Most modern calculators have keys that will compute ${}_nC_r$. The steps are practically the same as the computation for ${}_nP_r$.

Example 12–5 E illustrates counting combinations.

■ ***Example 12–5 E***

Determine the number of outcomes.

1. An individual has 12 (distinguishable) shirts and wants to pack 3 of them for a trip. In how many different ways can this be done?

We are interested in how many 3-element subsets of 12 elements there are; we do not care about the order in which the shirts are selected. Thus, we want ${}_{12}C_3 = \frac{12!}{3!9!} = \frac{12 \cdot 11 \cdot 10 \cdot 9!}{3!9!} = \frac{12 \cdot 11 \cdot 10}{6} = 220$.

2. On a ten-question test, in how many ways can a student get exactly seven questions correct (assuming that all questions are either right or wrong).

We want to know: out of ten elements, how many ways can we choose exactly seven of them. This is ${}_{10}C_7 = \frac{10!}{7!3!} = 120$. ■

In part 2 of example 12–5 E, if we asked instead how many ways can a student get exactly three questions out of ten wrong, which is of course the same as getting seven questions correct, this would be ${}_{10}C_3 = \frac{10!}{3!7!} = 120$. This illustrates the fact that ${}_nC_r = {}_nC_{n-r}$, which can be demonstrated as follows.

■ ***Example 12–5 F***

Show that $\binom{n}{n-r} = \binom{n}{r}$.

$$\binom{n}{n-r} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$ ■

[15]Introduced in section 12–3.

Further counting problems

Many counting problems cannot be answered with just one counting property. The example 12–5 G illustrates using more than one of the properties we have examined.

■ ***Example 12–5 G***

1. A class contains 30 students, 18 females and 12 males. How many different committees of 7 students can be formed if there must be 4 females and 3 males on the committee?

It is surprising how often looking at a problem a certain way can help. In this problem, we first simply ask how many committees of 4 females can be formed, and then how many committees of 3 males. These figures are ${}_{18}C_4 = 3{,}060$ and ${}_{12}C_3 = 220$. Now use the multiplication principle, because for each of the female committees we can choose any one of the male committees. The result is $3{,}060 \cdot 220 = 673{,}200$ committees.

2. How many different 7-card hands from a standard deck of 52 cards[16] are possible if the hand is to contain 3 hearts, 2 diamonds, and 2 cards that are not a heart or a diamond?

There is a great deal of similarity between this problem and the previous one. The class was divided into 2 categories (male and female), and the deck of cards is divided into 4 categories (13 of each of the four suits—hearts, diamonds, clubs, and spades). A card hand can be viewed as one committee. Thus, we proceed in the same manner, and first ask how many hands there are of 3 hearts (${}_{13}C_3 = 286$), how many hands there are of 2 diamonds (${}_{13}C_2 = 78$), and how many hands there are of black cards (clubs and spades) (${}_{26}C_2 = 325$). Now, since we do not care about the order in which cards are selected we can use the multiplication-of-choices property: $286 \cdot 78 \cdot 325 = 7{,}250{,}100$ such hands.

3. A race is to be run between two stables. The Adams stable will enter three of its seven horses, and the Baker stable will enter three of its eight horses. In how many ways can six horses finish the race?

The multiplication-of-choices principle tells us that the six horses to enter the race can be selected in ${}_7C_3 \cdot {}_8C_3 = 35 \cdot 56 = 1{,}960$ ways. However, the order in which the horses finish is important! For each of the 1,960 ways in which the race can start, there are $6! = 720$ ways (${}_6P_6$) for it to finish. Thus, there are $1{,}960 \cdot 720 = 1{,}411{,}200$ ways for the horses to be selected and then finish the race. ■

[16] A standard deck of cards has four suits, the two red suits (diamonds and hearts) and the two black suits (clubs and spades). Each suit has 13 cards: an ace, the numbered cards 2, 3, 4, 5, 6, 7, 8, 9, 10, and three face cards, jack, queen, king.

Mastery points

Can you

- Use the multiplication-of-choices property to solve counting problems?
- Compute and use permutations to solve counting problems?
- Compute and use combinations to solve counting problems?
- Combine these counting methods to solve problems?

Exercise 12–5

See figures 12–3 and 12–4 for the following problems.

1. There are 3 people in a race, A, B, and C.
   a. Draw a tree diagram of the ways in which the race can be finished.
   b. List all possible ways in which the race can be finished (for example, CBA).
2. A die is to be thrown twice.
   a. Draw a tree diagram of the ways in which the numbers on the die can appear.
   b. List all possible sequences of numbers that can occur (for example, 5, 2 if 5 appears on the first throw and 2 on the second).
3. A person is flipping a coin and noting heads (H) or tails (T) each time. The person does this three times.
   a. Draw a tree diagram of this experiment.
   b. List all possible outcomes (for example, HHT for head-head-tail).
4. An individual is tossing a die and flipping a coin.
   a. Draw a tree diagram in which the first branch reflects what happens with the coin (head/tail).
   b. Draw a tree diagram in which the first branch reflects what happens with the die (1/2/3/4/5/6).
   c. List all possible outcomes of this experiment. A typical outcome is H4, for heads on the coin and 4 on the die. This could just as well be described as 4H, since we are merely listing outcomes, not ways in which the outcome can be achieved.

Solve the following problems.

5. An individual has four pairs of slacks and six shirts. How many different combinations of shirts and slacks can this person wear?
6. A certain shoe store customer complained that a good shoe store would stock every style in every color and size. If the store has 65 styles and each style comes in 12 sizes, 2 widths, and 3 colors, how many different pairs of shoes would the store have to stock to have one of each? Would it be a good idea for a store to order shoes this way?
7. The Ahmes[17] (or Rhind) Papyrus is an ancient Egyptian papyrus. It contains a variation of the following problem. There are seven buildings for storing grain; each building is guarded by seven cats, each of which eats seven mice. If it were not for the cats, each of these mice would eat seven ears of corn, each of which could produce seven bushels of corn. How many bushels of grain are saved by the cats?
8. $\boxed{8.}$ A certain building has 12 entrances. In how many different ways can someone go in one entrance and out a different one?
9. A class has 12 females and 15 males. A female and a male are to be selected as student government representatives. In how many different ways can the selection be made?
10. A menu offers a choice of 5 appetizers, 3 salads, 12 entrees, 4 kinds of potatoes and 5 vegetables. A meal consists of one of each. In how many ways can a person select a meal?
11. In how many different ways can a student answer all the questions on a quiz consisting of 8 true or false questions?
12. From a standard deck of playing cards, in how many ways can a person select one heart, one club, one diamond, and one spade?

[17]Ahmes was the Egyptian scribe who copied this papyrus in about 1650 B.C.

Evaluate each expression.

13. $7!$ **14.** $9!$ **15.** $\frac{10!}{5!}$ **16.** $\frac{12!}{6!}$ **17.** $\frac{12!}{3!9!}$ **18.** $\frac{20!}{17!3!}$

19. ${}_6P_4$ **20.** ${}_8P_3$ **21.** ${}_6P_6$ **22.** ${}_{15}P_5$ **23.** ${}_{18}P_6$ **24.** ${}_{20}P_5$

25. ${}_{20}P_1$ **26.** ${}_4P_3$ **27.** Show that ${}_nP_n = n!$. **28.** Show that ${}_nP_1 = n$.

Solve the following problems.

29. In a nine-horse race, how many different first-second-third place finishes are possible?

30. A president, vice-president and secretary are to be elected from a club with 25 members. In how many different ways can these offices be filled?

31. A basketball team has 15 players. In how many ways can a captain and cocaptain be chosen from the players?

32. In how many different ways can eight students be seated in a row of eight chairs?

33. In how many ways can seven books be arranged on a shelf?

34. In horse racing, a perfecta bet picks the first place finisher in a race and the second place finisher. If a race has 11 horses, how many perfecta bets are possible?

35. In horse racing, a trifecta bet picks the first, second, and third place finishers in a race. If a race has 11 horses, how many trifecta bets are possible?

36. A contractor wishes to build eight houses, all different in design. In how many ways can these houses be placed if there are five lots on one side of the street and three lots on the other side?

37. There are 15 players on a baseball team. Nine are selected for the starting lineup, and they are given a certain batting order. We will call this the initial batting order. How many different initial batting orders are possible?

38. A football coach checks a player for six performance traits and lists the three strongest on a card; the coach lists these three strongest traits in order of the player's proficiency. How many different evaluations are possible?

39. A neighborhood children's club's secret password is ''zyzzybalubah.'' How many different 12-letter ''words'' can be formed using all the letters of the secret word?

40. How many different words can be formed using all the letters of the word ''mammal''?

41. In how many different ways can the monomial $3a^2b^4c^5$ be written without using exponents? (One way is ''abc(3)abbbcccc.'')

42. In how many different ways can the monomial $3a^2b^4c^5$ be written without using exponents, if the numerical coefficient must appear first? (See problem 41.)

Evaluate each expression.

43. ${}_{15}C_{10}$ **44.** ${}_{20}C_6$ **45.** ${}_8C_5$ **46.** ${}_{18}C_{16}$

47. ${}_{14}C_2$ **48.** ${}_{14}C_{12}$ **49.** Show that ${}_nC_n = 1$. **50.** Show that ${}_nC_1 = n$.

51. A child is to be allowed to choose four different candies from a box containing 20 different kinds. How many different selections are possible?

52. How many different five-card hands can be dealt from a standard 52-card deck of playing cards?

53. On an examination consisting of 12 essay questions the student may omit any 4. In how many different ways can the student select the problems to be answered (the order of selection is not important)?

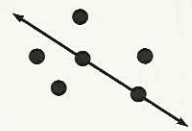
54. Suppose NASA has 19 astronauts suitable for the next space mission. A crew consists of 5 astronauts. How many different crews are possible?

55. A pizza parlor has eight different toppings for its pizzas. A regular pizza has any two of these toppings (they must be different). How many combinations of toppings are there for a regular pizza?

56. Ten individuals want to form two teams of five players each. In how many different ways can this be done?

57. Seven distinct points lie on a circle. How many different inscribed triangles can be drawn such that all of their vertices come from these points?

58. How many different straight lines can be drawn through the six points shown in the figure? (One such line is shown.) No three points are collinear.



The following problems may require the multiplication-of-choices property, permutations, and/or combinations.

59. Fourteen children are playing a game of musical chairs. If there is a row of ten chairs in which the children can sit when the music stops, how many different groups of four children could be eliminated when the music stops?

60. From a standard deck of playing cards how many ways can a person select an ace, a king, a queen, and a jack (without regard to the order of selection)?

61. Suppose there are 17 players on a baseball team.

- **a.** In how many different ways can a team of 9 be chosen if every player can play every position?
- **b.** In how many different ways can a captain and a co-captain be chosen (assuming the cocaptain and captain are different positions)?
- **c.** How many different batting orders are possible (considering all possible 9-player teams and all possible batting orders for 9 players)?
- **d.** In how many different ways can the team membership be reduced to 12 players?

62. **a.** How many ways can three males and four females sit in a row?

- **b.** How many ways can three males and four females sit in a row if males and females must alternate?
- **c.** How many arrangements of males and females are possible (i.e., permutations in which males are indistinguishable and females are indistinguishable)?

63. In how many different ways can a student answer all the questions on a quiz consisting of ten true/false questions?

64. Selecting from the set of digits $\{1, 2, 3, 4, 5\}$ how many of the following are possible?

- **a.** Three-digit numbers
- **b.** Three-digit odd numbers
- **c.** Three-digit numbers, where the first and last digit must be even
- **d.** Three-digit numbers using only odd digits

65. Answer problem 64 if repetition of a digit is not allowed.

The following problems may involve several counting properties to solve.

66. In how many ways can a group of eight males and six females be divided into two groups consisting of four males and four females?

67. Ten teams are in a league. If each team is required to play every other team twice during the season, what is the total number of league games that will be played?

68. A shopper is choosing 6 different frozen dinners from a selection of 17 and 4 different fruits from a selection of 11. In how many different ways can the selections be made?

69. If a group consists of 18 men and 12 women, in how many different ways can a committee of 6 be selected if:

- **a.** the committee is to have an equal number of men and women?
- **b.** the committee is to be all women?
- **c.** there are no restrictions on membership on the committee?

70. At the beginning and end of every meeting of a certain club, each member must give the ritual handshake to every other member. If there are 20 members present at the meeting, how many different handshakes will take place?

71. In a certain computer there are 256 CPUs (central processor units); each is connected to each of the others. How many such connections are there?

72. A test contains three groups of questions, A, B, and C, that contain five, four, and three questions, respectively. If a student must select three questions from group A and two from each of the remaining groups, how many different tests are possible?

73. In horse racing, a double trifecta bet picks the first, second, and third place finishers in the first two races. If there are nine horses in the first race and eight horses in the second, how many different bets are possible?

74. Show that $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$.

75. Show that $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$.

$\boxed{76.}$ Show that $\binom{n}{r+1} + \binom{n}{r} = \binom{n+1}{r+1}$.

77. We defined ${}_nP_r = n(n-1)(n-2)\cdots(n-[r-1])$. Show that ${}_nP_r = \dfrac{n!}{(n-r)!}$ gives the same result for $n = 10$ and $r = 3$.

78. We have two ways to compute ${}_nP_r$:

$${}_nP_r = n(n-1)(n-2)\cdots(n-[r-1])$$

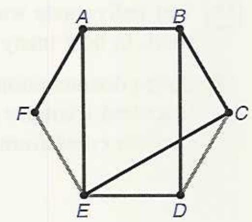
$${}_nP_r = \frac{n!}{(n-r)!}$$

Consider a calculator that has only a factorial $\boxed{x!}$ key and consider computing ${}_{300}P_3$ and ${}_{30}P_{28}$ to find an advantage that each method has over the other.

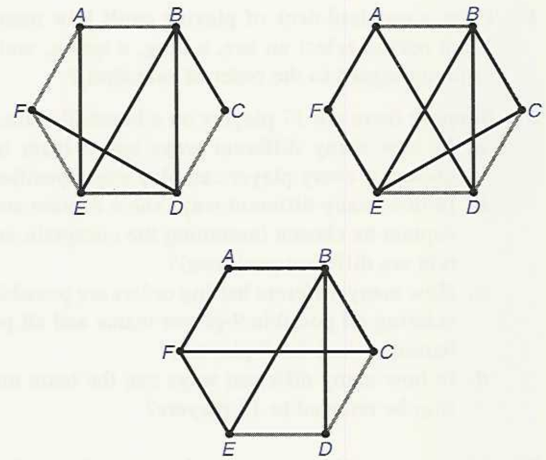
79. Show that $\dfrac{{}_nP_r}{r!} = \dfrac{n!}{r!(n-r)!}$ is true for $n = 10$ and $r = 3$.

$\boxed{80.}$ The "party puzzle" states the following. Choose any six people at a party (or anywhere else). Then one of the following statements is true. Either there is a group of three who all know one another, or there is a group of three in which none of the members knows either of the other two.[18]

For example, the diagram shows six people, A, B, C, D, E, and F. The solid lines show who knows whom. Thus, person A knows persons F and E, and person D only knows person B. In this case, persons B, E, and F form a group of three, none of whom knows the other two.



a. In each of the three situations shown in the three-part figure, find a group of three people who either mutually know each other or who mutually do not know each other.



b. Using the diagrams as shown simplifies solving this problem. However, if we were to check each possible group of three people in a given situation the work would be tedious. How many groups of three people are there, given six people?

[18]A proof that one of these two types of groups can always be found is in the article "Ramsey Theory," by Ronald L. Graham and Joel H. Spencer, *Scientific American,* July 1990.

Skill and review

1. Find $\sum_{i=1}^{\infty} (\frac{1}{2})^i$.

2. Find $2 + 7 + 12 + 17 + \cdots + 97$.

3. Solve $\left|\dfrac{2-3x}{4}\right| \le 10$.

4. Graph $f(x) = \dfrac{x^2-1}{x^2-4}$.

5. Graph $f(x) = (x-2)^2(x+1)^3$.

6. Solve $x + \dfrac{1}{x} = 3$.

7. Solve $\dfrac{x+y}{x} = \dfrac{x-y}{2}$ for y.

12–6 Introduction to probability

A certain test for determining whether or not an individual has used certain drugs is 90% accurate. That is, it will detect 90% of those drug users who are tested. The test also gives 10% false positives. That is, of those tested who do *not* use drugs the test will falsely report that the individual does use them. Suppose that 100,000 workers are to be tested for drug use, and that 15% of these workers use the drugs in question. If the test reports that an individual uses the drugs, what is the probability that this is *not* true?

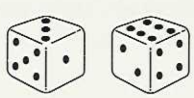
This problem is just one of the many places where probability occurs every day in society. Probability is the subject of this section.

Terminology

Probability is a means of measuring uncertainty. The theory of probability is said to have begun in 1654 when a French nobleman and gambler, the Chevalier de Méré, asked his friend, the famous French mathematician-philosopher-writer Blaise Pascal, questions about gambling. Pascal's attempt to answer the Chevalier's questions, along with correspondence between Pascal and the French mathematician Pierre de Fermat, began the modern theory of probability. Today probability is used to explain atomic physics and human behavior; to derive the charge for an insurance policy; to decide the proper medical treatment for a disease; and to predict the chance that it will rain tomorrow, that a traffic light will last at least 1 year without failure, or the probability that an engine on a multiengine jet will fail on an overseas flight.

In the study of probability, any happening whose result is uncertain is called an **experiment.** The different possible results of the experiment are called **outcomes,** and the set of all possible outcomes of an experiment is called the **sample space** of the experiment, denoted by S. In this text all sample spaces are finite—that is, they have a limited number of outcomes.

An **event** is a subset of the sample space. If an event is the empty set it is said to be **impossible.** If an event has only one element it is called a **simple event.** Any nonempty event that is not simple is called a **compound event.**



To illustrate, consider the *experiment* of rolling two dice.[19] When the first die is rolled, any value from 1 through 6 can appear, and the same is true for the second die. Let S be all the possible outcomes of rolling the two dice (the *sample space*). Let A be the *event* of rolling a total of 7, and let B be the *event* of rolling a total of 2. Figure 12–5 illustrates S, A, and B, where S is all of the number pairs. For example, (1,3) represents getting a 1 on the first die, and a 3 on the second.

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Figure 12–5

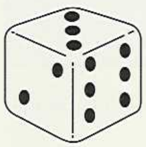
[19]A standard die has six sides. Each side has a different number of from one through six dots. Whenever we refer to a die it will mean a standard die.

The event A is the set of two-tuples in which the sum of the elements is 7. The event $A = \{(6,1), (5,2), (4,3), (3,4), (2,5), (1,6)\}$. A is a *compound* event. The event B is the set $B = \{(1,1)\}$. B is a *simple* event.

Example 12–6 A illustrates how to determine a sample space.

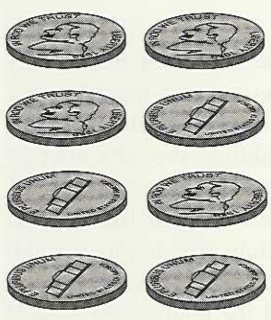
■ ***Example 12–6 A***

Determine the sample space for each experiment.



1. Tossing a six-sided die.
The sample space consists of six outcomes, which are represented by the integers from 1 to 6.

$$S = \{1, 2, 3, 4, 5, 6\}$$



2. Flipping a coin twice.
The possible outcomes are two heads, head, then tail, tail, then head, two tails.
If we use H for a head and T for a tail, the sample space would be

$$S = \{\text{HH, HT, TH, TT}\}.$$

In this sample space the event of a head and a tail can occur in two ways, HT and TH, and is a compound event. The event of getting two heads can occur only one way, and is a simple event. ■

Probability of an event

If we wish to determine the probability of an event, we must know how many outcomes make up the event and how many outcomes make up the sample space. The number of outcomes in event A is represented by $n(A)$, and the number of outcomes in the sample space is represented by $n(S)$. If the outcomes of a sample space are *equally likely,* then every outcome in the sample space has the same chance of occurring. We will concern ourselves only with equally likely events.

> **Probability of an event**
> If an event A is made up of $n(A)$ equally likely outcomes from a sample space S that has $n(S)$ equally likely outcomes, then the probability of the event A, represented by $P(A)$, is
> $$P(A) = \frac{n(A)}{n(S)}$$

We know that $0 \le n(A)$ and $0 \le n(S)$, because each value represents counting a number of events. Also, A represents a subset of S, so $n(A) \le n(S)$. Thus,

$$0 \le n(A) \le n(S)$$

Dividing each member by $n(S)$, when $n(S) > 0$, we obtain

$$\frac{0}{n(S)} \leq \frac{n(A)}{n(S)} \leq \frac{n(S)}{n(S)}$$

or

$$0 \leq P(A) \leq 1$$

The following summarizes these and several other principles of probability.

Basic probability principles

If the event A has $n(A)$ equally likely outcomes and the sample space S has $n(S)$ equally likely outcomes, and $n(S) > 0$, then

1. $0 \leq P(A) \leq 1$ 2. $P(A) = \dfrac{n(A)}{n(S)}$
3. $P(A) = 0$ means event A cannot occur, and is called an impossible event.
4. $P(A) = 1$ means event A must occur and is called a certain event.

Example 12–6 B illustrates the terminology and values associated with the probability of an event.

■ *Example 12–6 B*

When rolling a single six-sided die, name the following events and give their probability.

Event	Name	Probability
1. $\{1\}$	Rolling a one	$\frac{1}{6}$
2. $\{2, 4, 6\}$	Rolling an even number	$\frac{3}{6}$ or $\frac{1}{2}$
3. $\{7\}$	Rolling a 7 (impossible event)	0
4. $\{1, 2, 3, 4, 5, 6\}$	Rolling a 1, 2, 3, 4, 5, or 6	1

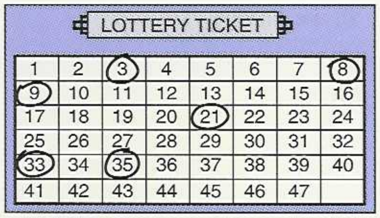
■

In the examples of rolling a single die or flipping a coin twice, it was easy to list all of the possible outcomes in the sample space. When the problem has a large number of possible outcomes, we will not always try to list all possible outcomes. Instead we will use the counting techniques seen in section 12–5 to determine the number of outcomes that make up an event or the sample space. Example 12–6 C illustrates finding probabilities in which the sample space has a large number of elements.

■ *Example 12–6 C*

Find the probability of the given event.

1. A coin is flipped 3 times. What is the probability of exactly two heads? The sample space is $S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$. The event of getting exactly two heads is $A = \{\text{HHT, HTH, THH}\}$. Thus, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{3}{8}$.



2. To win the State of Michigan lottery a person must correctly select the six integers drawn by the lottery commission from the set of integers from 1 to 47. If a person buys one ticket, what is the probability of winning?

The sample space is the set of all collections of 6 different numbers chosen from 1 through 47. By way of example, the figure shows the selection of 3, 8, 9, 21, 33, and 35. The order of selection of the integers is not important, so the size of the sample space is the number of combinations of size six, chosen from 47 available items, ${}_{47}C_6$. Thus,

$$n(S) = {}_{47}C_6 = \frac{47!}{41!\,6!} = 10,737,573.$$

The event A of winning is one of these selections, so $n(A) = 1$.

Thus, the probability of winning the lottery $P(A)$ is $P(A) = \dfrac{n(A)}{n(S)}$

$$= \frac{1}{10{,}737{,}573} \approx 0.000000093.$$

Note In a calculator the display may look like $\boxed{9.313091515^{-08}}$. This is in scientific notation and means approximately $9.3 \times 10^{-8} \approx 0.000000093$. (See section 1–2 to review scientific notation.)

3. Five cards are selected from a standard pack of playing cards.[20] What is the probability that all five cards are hearts?

The sample space S is the set of all possible five-card hands chosen from 52 cards. The event A is the set of all possible five-card hands chosen from the 13 hearts.

$$P(A) = \frac{n(A)}{n(S)} = \frac{{}_{13}C_5}{{}_{52}C_5} = \frac{1{,}287}{2{,}598{,}960} = \frac{33}{66{,}640} \approx 0.00050$$

4. A congressional committee contains eight Democrats and six Republicans. A subcommittee of four people is randomly chosen. What is the probability that all four people will be Republicans?

The sample space S is the set of all four-person committees that can be selected from 14 people. The event R of an all-Republican committee is the number of four-person committees that can be chosen from the 6 Republicans.

$$P(R) = \frac{n(R)}{n(S)} = \frac{{}_6C_4}{{}_{14}C_4} = \frac{15}{1{,}001} \approx 0.015$$

Mutually exclusive events

If two events from the same sample space have no outcomes in common, they are called **mutually exclusive events.** For example, if a single die is tossed, the event A of rolling a number less than 3 ($A = \{1,2\}$) and the event B of rolling a number greater than 4 ($B = \{5,6\}$) are mutually exclusive. In this situation the probability of the compound event of A or B is the sum of $P(A)$ and $P(B)$.

[20]See footnote 16.

Probability of mutually exclusive events

If the events A and B are mutually exclusive, then the probability that A or B will occur is

$$P(A \text{ or } B) = P(A) + P(B)$$

Example 12–6 D illustrates finding probabilities of compound events which are composed of mutually exclusive events.

■ ***Example 12–6 D***

Find the probability of the given event.

1. A card is drawn from a standard deck of 52 cards. What is the probability of an ace or a jack?

   The card that is drawn cannot be an ace and a jack at the same time. Therefore, the two events are mutually exclusive, and therefore the probability of one or the other is the sum of the probability of each individually. If $P(A)$ is the probability of an ace and $P(J)$ is the probability of a jack, we have

$$P(A) = \frac{\text{number of aces}}{\text{number of cards in deck}} = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

$$P(J) = \frac{\text{number of jacks}}{\text{number of cards in deck}} = \frac{n(J)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

$$P(A \text{ or } J) = P(A) + P(J) = \tfrac{1}{13} + \tfrac{1}{13} = \tfrac{2}{13}$$

2. A single die is tossed. What is the probability of a 5 or an even number?

   Since 5 is not even, the two events are mutually exclusive. Let F be the event of a 5: $F = \{5\}$. Then $P(F) = \dfrac{n(F)}{n(S)} = \tfrac{1}{6}$. Let E be the event of an even number: $E = \{2, 4, 6\}$. Then $P(E) = \dfrac{n(E)}{n(S)} = \tfrac{3}{6} = \tfrac{1}{2}$. Therefore,

   $P(F \text{ or } E) = P(F) + P(E) = \tfrac{1}{6} + \tfrac{1}{2} = \tfrac{2}{3}$. ■

Events that are not mutually exclusive

Consider the problem where we are asked to find the probability, when rolling a single die, of an even number or a number greater than 4. These two events could be described as $E = \{2, 4, 6\}$ and $G = \{5,6\}$. These events are not mutually exclusive, because the outcome of a 6 is part of both events.

To count the probability of the event E or G we cannot simply add the probability of each event separately. For example, calculating as we do for mutually exclusive events

$$P(E \text{ or } G) = P(E) + P(G) = \tfrac{3}{6} + \tfrac{2}{6} = \tfrac{5}{6}$$

is incorrect. The event E or G is the set $\{2, 4, 5, 6\}$, and $P(E \text{ or } G) = \tfrac{4}{6}$. Thus $\tfrac{5}{6}$ is too large by $\tfrac{1}{6}$.

The problem is that the simple event of rolling a 6 is in both compound events E and G. Thus, it was counted twice instead of once. We can adjust the method for calculating probabilities of mutually exclusive events by subtracting those probabilities that are counted twice. This leads to the following conclusion.

General addition rule of probability

If the events A and B are from the same sample space, then the probability that A or B will occur is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Note If the events A and B are mutually exclusive they cannot occur at the same time, and $P(A \text{ and } B) = 0$.

Now we can calculate the probability of the event E or G (above) correctly. First, note that the events E and G is the event $\{6\}$, since this is the only event both in the set E and G. Thus, $P(E \text{ and } G) = \frac{1}{6}$.

$$P(E \text{ or } G) = P(E) + P(G) - P(E \text{ and } G) = \tfrac{3}{6} + \tfrac{2}{6} - \tfrac{1}{6} = \tfrac{4}{6} = \tfrac{2}{3}$$

These ideas are further illustrated in example 12–6 E.

■ ***Example 12–6 E***

1. A card is drawn from a standard deck of 52 cards. What is the probability of a diamond or a face card?

Let D be the event of a diamond and F be the event of a face card. The events D and F are not mutually exclusive, since they have the three cards king, queen, and jack of diamonds in common. We find the probability of the event D or F as follows:

$$P(D) = \frac{\text{number of diamonds}}{\text{number of cards in deck}} = \frac{n(D)}{n(S)} = \frac{13}{52}$$

$$P(F) = \frac{\text{number of face cards}}{\text{number of cards in deck}} = \frac{n(F)}{n(S)} = \frac{12}{52}$$ 4 face cards per suit

$$P(D \text{ and } F) = \frac{\text{number of face cards that are diamonds}}{\text{number of cards in deck}} = \frac{n(D \text{ and } F)}{n(S)} = \frac{3}{52}$$

Therefore,

$$P(D \text{ or } F) = P(D) + P(F) - P(D \text{ and } F) = \tfrac{13}{52} + \tfrac{12}{52} - \tfrac{3}{52} = \tfrac{22}{52} = \tfrac{11}{26}$$ ■

Complement of an event

The **complement of an event** A is the set of all outcomes in the sample space that are not part of the event A. We denote the complement of an event A by A', which we read as "A's complement." By definition, A and A' are mutually exclusive events. The following can be seen to be true.

$n(A \text{ or } A') = n(S)$
$P(A \text{ or } A') = P(S) = 1$
$P(A \text{ and } A') = 0$ (since A and A' cannot happen at the same time).

Using the general addition rule of probability, we obtain

$$P(A \text{ or } A') = P(A) + P(A') - P(A \text{ and } A')$$
$$1 = P(A) + P(A') - 0 \quad \text{Substitute values from above}$$
$$1 - P(A) = P(A')$$

Thus we can state the following.

Probability of complementary events

If A is any event and A' is its complement, then

$$P(A') = 1 - P(A)$$

Example 12–6 F shows how to find the probability of an event by first finding the probability of the complementary event.

Example 12–6 F

Find the probability of the given event.

1. A card is drawn from a standard deck of 52 cards. What is the probability of not selecting a face card?

   The face cards are the kings, queens, and jacks of the four suits. Thus there are 12 face cards. Let F be the event of drawing a face card. Then $P(F) = \frac{12}{52} = \frac{3}{13}$. F' is the event of drawing a card which is not a face card. Then,

   $$P(F') = 1 - P(F) = 1 - \tfrac{3}{13} = \tfrac{10}{13}$$

2. Five cards are selected from a standard deck of cards. What is the probability that one or more of the cards is not a heart?

   If H is the event of drawing five hearts, then H' is the event of drawing five cards in which at least one is not a heart.

   $$P(H') = 1 - P(H) = 1 - \frac{{}_{13}C_5}{{}_{52}C_5} = \frac{66{,}640}{66{,}640} - \frac{33}{66{,}640} = \frac{66{,}607}{66{,}640} \approx 0.9995$$

Mastery points

Can you

- Determine the number of outcomes in an event?
- Determine the number of outcomes in a sample space?
- Determine the probability of an event?

Exercise 12–6

A coin is tossed two times. Find the probability of

1. exactly one head.
2. one head and one tail.
3. at least one head.
4. no heads.

A coin is tossed three times. Find the probability of

5. all tails.
6. no tails.
7. at least two tails.
8. at most two tails.

A coin is tossed four times. Find the probability of

9. exactly two heads. **10.** exactly two tails. **11.** less than two heads. **12.** more than two heads.

A card is drawn from a standard deck of playing cards. What is the probability of

13. a ten? **14.** a seven? **15.** a club? **16.** a heart?
17. a card from 4 through 9, inclusive? **18.** a card from 3 through 6, inclusive?
19. a red 7? **20.** a black 8? **21.** a black 4 or 5? **22.** a red king?

A card is drawn from a standard deck of playing cards. Find the probability that the card is

23. a heart or a 7.
24. a diamond or a queen.
25. from 2 through 6, inclusive, or a spade.
26. from 5 through 8, inclusive, or a club.
27. not a 10.
28. not a jack.
29. not from 4 through 10, inclusive.
30. not from 2 through 5, inclusive.
31. not a club.
32. not a heart.
33. not red.
34. not black.

A roulette wheel contains the numbers from 1 through 36. Eighteen of these numbers are red, and the other 18 are black. There are two more numbers, 0 and 00, which are green. The wheel is spun, and a ball allowed to fall on one of these 38 locations at random, as the wheel stops. What is the probability that the ball will land on

35. the 10? **36.** a red number? **37.** a number that is not green?
38. a black or green number? **39.** a white number? **40.** an odd number?
41. an even, nonzero number?

A bowl contains 24 balls. Six are red, 10 are blue, and 8 are white. If one ball is randomly selected, what is the probability that the ball is

42. red? **43.** red or white? **44.** red, white, or blue?
45. black? **46.** not white? **47.** not red or white?

Find the probability of the given event. You may need to use some of the counting techniques of section 12–5.
An automobile parts supplier has 18 alternators on hand of a certain model. Ten of the alternators are new, and 8 are remanufactured. If 6 of the alternators are chosen randomly for a shipment, what is the probability that the shipment

48. contains all remanufactured alternators?
49. contains all new alternators?
50. contains half new and half remanufactured alternators?
51. contains 2 new and 4 remanufactured alternators?

Five cards are drawn from a standard deck of playing cards. Find the probability of each event.

52. All five cards are spades. **53.** All five cards are red. **54.** All five cards are black.
55. None of the cards are clubs. **56.** None of the cards are face cards. **57.** All of the cards are face cards.
58. Three black and two red cards. **59.** One black and four red cards. **60.** Three clubs and two hearts.
61. Four diamonds and one spade.

62. In a certain state lottery, 6 numbers must be chosen correctly from 46 numbers. What is the probability of making this choice correctly?

63. In a minilottery you must choose 2 numbers correctly from the numbers 1 through 15. What is the chance of doing this successfully?

64. A certain state instant-winner game has ten circles. Two of the circles cover the word WIN and eight cover the word SORRY. The game is played by scratching the cover off of two (and only two) of the circles. If both reveal the word WIN the player wins. What is the probability of winning this game?

65. A certain assembly line built 50 television sets today. Of those, 4 are defective.

a. If 1 of the TVs is chosen at random, what is the probability that it is defective?

b. If a shipment of 6 TVs was sent out before testing, what is the probability that at least 1 of these was defective?

66. A hospital has 100 units of type A blood on hand. Of those, 3 are infected with hepatitis.

a. If a patient receives 1 unit of this blood, what is the probability of getting a unit that is infected?

b. If a patient receives 4 units of this blood, what is the probability of not getting any units that are infected?

A doctor has four patients waiting, patients A, B, C, and D. They do not have appointments, and arrived at the same time. If the doctor chooses the order in which to see the patients at random, what is the probability that

67. B will be chosen first?

68. A will be chosen second?

69. A and B are seen before C or D?

70. C is seen before D?

Apply the following to problems 71–76. The idea of probability goes far beyond that presented in this section. By way of example, consider the formula $P(k,t) = \dfrac{e^{-n} \times n^k}{k!}$, where $n = \dfrac{t}{\text{MTBF}}$, and e is the constant introduced in section 9–4. This formula gives the probability of k failures in time period t, in a situation in which something, say an aircraft engine, fails based on what is called a normal distribution. MTBF stands for mean time between failures, and is the average time between failures. For example, if an aircraft engine is expected to operate 2,000 hours between failures (MTBF = 2,000), then the probability of two failures in 500 hours would be found by using $n = \dfrac{500}{2{,}000} = 0.25$.

$$P(2, 500) = \frac{e^{-0.25} \cdot 0.25^2}{2!} \approx 0.024$$

71. Suppose a computer's MTBF is 4,000 hours. Find the probability of one failure in 3,000 hours of operation.

72. Suppose a computer's MTBF is 1,000 hours of operation. Find the probability of two failures in 3,000 hours of operation.

73. A computer's MTBF is 2,000 hours. Find the probability of at least one failure in 3,000 hours of operation. (This is 1, less the probability of no failures in 3,000 hours of operation.)

74. A computer's MTBF is 1,000 hours of operation. Find the probability of at least two failures in 3,000 hours of operation. (This is 1 less the sum of the probabilities of 0 and 1 failures.)

75. Suppose an automobile's alternator has a MTBF of 1,000 hours. Find the probability of at least 1 failure in 1,500 hours of operation. (1, less the probability of no failures.)

76. Referring to problem 75, find the probability of no failures in 800 hours of operation.

77. A certain test for determining whether or not an individual has used certain drugs is 90% accurate. That is, it will detect 90% of those drug users who are tested. The test also gives 10% false positives. That is, of those tested who do *not* use drugs the test will falsely report that the individual does use them. Suppose that 100,000 workers are to be tested for drug use, and that 15% of these workers use the drugs in question. If the test reports that an individual uses the drugs, what is the probability that this is *not* true?

78. The current test for the HIV (AIDS) virus (called ELISA) has sensitivity 0.983, and specificity 0.998. Sensitivity means the probability that a person with the virus will test positive. Specificity means the probability that a person who does not have the virus will test negative. If 500 people out of 1 million carry the virus, and those million are tested with the ELISA test, what is the probability that a person who tests positive actually has the virus? (In practice, those who test positive are given another test, which is much more accurate but much more expensive. Even the second test is still not perfect, however.)

Skill and review

1. Solve $S = \frac{1}{2}[a - b(a + c)]$ for a.
2. If $f(x) = x^3 - x^2 - x$, compute $f(a - 1)$.
3. Find the inverse function of $f(x) = \dfrac{1 - 2x}{3}$.
4. Find $\log_4 64$.
5. Solve $\log(x + 3) + \log(x - 1) = 1$.
6. Graph $f(x) = \sqrt{x - 3} - 1$.

12–7 Recursive definitions and recurrence relations—optional

The Fibonacci sequence is the sequence 1, 1, 2, 3, 5, 8, . . . , where each element after the first two is the sum of the two previous elements. Find a way to compute the value of any given element without having to compute all previous values.

Recursive definitions

In addition to advanced mathematics, the material in this section is used in computer science.[21] It is not common in most other areas where mathematics is applied—this is why this section is optional.

In section 12–1 we discussed sequences that are defined by an expression or rule for the nth term. For example, $a_n = 2n - 3$ defines a sequence by an expression for the nth term. If we define another sequence as the digits in the decimal expansion of $\sqrt{2}$, we specify a rule for the sequence without specific reference to the nth term.

Another way to define the nth term of a sequence is to give a rule that depends on previously known terms; this is called a **recursive definition.** An example of this would be the rule $a_n = \begin{cases} 3 \text{ if } n = 0 \\ 2a_{n-1} \text{ if } n > 0 \end{cases}$. We are indexing the sequence beginning at 0 instead of 1 because this makes some of the following discussion simpler. This rule says that the first value in the sequence is 3, and each value after that is twice the previous value. Thus, this rule gives the sequence 3, 6, 12, 24, 48, ... The part of the definition that is $2a_{n-1}$ is called its **recursive part,** and the part that is 3 is called the **terminal part.** Any recursive definition needs a terminal part, or there is no place to ''start.'' With this rule, to find, say, a_{50} we would have to first know a_{49}, and to find a_{49} we would have to know a_{48}, etc., until we arrived at the terminal part of the definition. This is a ''weakness'' of recursive definitions—you cannot just jump in any where you want!

It can be seen that the sequence 3, 6, 12, 24, 48, . . . is a geometric sequence, with $a_0 = 3$ and $r = 2$, so we could formulate the rule[22] $a_n = 3(2^n)$,

[21]This chapter relies heavily on *Mathematics for the Analysis of Algorithms,* by Daniel Green and Donald Knuth, Birkhäuser, Boston, 1982. To see just how mathematical computer science can be see *The Art of Computer Programming,* a three volume tour de force by Donald Knuth. It is published by Addison-Wesley, Reading, Massachusetts.

[22]When we index a series from 0 instead of 1 the resulting formulas will be different from those presented earlier.

$n = 0, 1, 2, \ldots$, from which we could deduce that $a_{50} = 3(2^{50})$. This second definition is sometimes called a **closed form definition.** Thus, the following are two different rules for the same sequence.

$$a_n = \begin{cases} 3 \text{ if } n = 0 \\ 2a_{n-1} \text{ if } n > 0 \end{cases} \qquad \text{recursive}$$

$$a_n = 3(2^n),\ n = 0, 1, 2, \ldots \qquad \text{closed form}$$

For computational purposes, a closed form rule is more useful than a recursive rule, but there are times when the closed form rule is hard to find or simply does not exist. In computer science, many algorithms (procedures) are most easily described in a recursive manner. For example, the computation of factorials can be described recursively: if $n > 1$, then $n! = n \cdot (n - 1)!$. This can be translated into a one-line computer program in most modern programming languages. Another example is discovering the shortest path that a traveling salesperson could follow to visit every customer in a certain region once.

Example 12–7 A illustrates applying recursive definitions to list the elements of a sequence.

■ Example 12–7 A

Find the first five terms of each recursively defined sequence.

1. $a_n = \begin{cases} 2 \text{ if } n = 0 \\ a_{n-1} + 3 \text{ if } n > 0 \end{cases}$

$a_0 = 2$ — Terminal part
$a_1 = a_0 + 3 = 2 + 3 = 5$
$a_2 = a_1 + 3 = 5 + 5 = 8$
$a_3 = a_2 + 3 = 8 + 3 = 11$

Thus, the sequence is 2, 5, 8, 11, 14, . . . (an arithmetic sequence).

2. $a_n = \begin{cases} 1 \text{ if } n = 0 \\ 3a_{n-1} \text{ if } n > 0 \end{cases}$

$a_0 = 1$ — Terminal part
$a_1 = 3a_0 = 3(1) = 3$
$a_2 = 3a_1 = 3(3) = 9$

This is 1, 3, 9, 27, 81, . . . (a geometric sequence).

3. $a_n = \begin{cases} 1 \text{ if } n = 0,\ 3 \text{ if } n = 1 \\ 2a_{n-1} + 3a_{n-2} \text{ if } n > 1 \end{cases}$

In this sequence each term after the first two depends on the two previous terms.

$a_0 = 1,\ a_1 = 3$ — Terminal part
$a_2 = 2a_1 + 3a_0 = 2(3) + 3(1) = 9$
$a_3 = 2a_2 + 3a_1 = 2(9) + 3(3) = 27$
$a_4 = 2a_3 + 3a_2 = 2(27) + 3(9) = 81$

Thus, the first five terms of the sequence are 1, 3, 9, 27, 81, which appears to be the beginning of a geometric sequence.

4. $a_n = \begin{cases} 2 \text{ if } n = 0, 3 \text{ if } n = 1 \\ 2a_{n-1} + 3a_{n-2} \text{ if } n > 1 \end{cases}$

$$a_0 = 2, a_1 = 3$$
$$a_2 = 2a_1 + 3a_0 = 2(3) + 3(2) = 12$$
$$a_3 = 2a_2 + 3a_1 = 2(12) + 3(3) = 33$$
$$a_4 = 2a_3 + 3a_2 = 2(33) + 3(12) = 102$$

Thus, the sequence is 2, 3, 12, 33, 102, ... Observe that the recursive rule is the same for this sequence, but the first term is different, and the resulting sequence is not a geometric sequence.

5. $a_n = \begin{cases} 1 \text{ if } n = 0, 1 \text{ if } n = 1 \\ a_{n-1} + a_{n-2} \text{ if } n > 1 \end{cases} = 1, 1, 2, 3, 5, 8, \ldots$ ■

The sequence in part 5 of example 12–7 A, 1, 1, 2, 3, 5, 8, . . . , is called the **Fibonacci sequence.** Leonardo of Pisa, also called Fibonacci, described the sequence in a problem[23] in his book *Liber abaci* (book of the abacus) in 1202. This sequence actually occurs in solving certain problems in computer science.[24]

It is often important to have an expression for a_n in a sequence (that is to have a closed form definition of the sequence). This is not always possible, but it can be done for the sequences presented in example 12–7 A. Example 12–7 B shows how this is done when the sequence fits a pattern we saw in section 12–1.

■ *Example 12–7 B*

Find an expression for a_n for the sequences in parts 1 and 2 of example 12–7 A.

1. The sequence is an arithmetic sequence: 2, 5, 8, 11, . . . with $a_0 = 2$, $a_1 = 5$, and $d = 3$.

$$a_n = a_1 + (n - 1)d$$
$$a_n = 5 + 3(n - 1) = 3n + 2$$

General formula for *n*th term of an arithmetic sequence

Thus, $a_n = 3n + 2, n = 0, 1, 2, \ldots$.

2. 1, 3, 9, 27, 81 is a geometric sequence with $a_0 = 1$, $a_1 = 3$, and $r = 3$.

$$a_n = a_1 r^{n-1}$$
$$a_n = 3(3^{n-1}) = 3^n$$

General formula for *n*th term of a geometric sequence.

Thus, $a_n = 3^n, n = 0, 1, 2, \ldots$. ■

[23] Leonardo wrote "How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on?" His answer, under the conditions he gave, is 377.

[24] In fact, this sequence is found in fields as varied as biology and art. It is found in the seed pattern of the sunflower and in pine cones and is related to the dimensions of the chambers of the nautilus seashell. It has produced so much interest that there is a periodical called the *Fibonacci Quarterly,* produced by the Fibonacci Association.

Recurrence relations

To find an expression for a_n in the remaining three sequences of example 12–7 A, we resort to **recurrence relations.** We will illustrate finding recurrence relations for the case where the recursive definition expresses a_n as a linear combination of previous terms. The following procedure is used.

> **Recurrence relations**
>
> A procedure for finding expressions for a_n for recursively defined sequences where the recursive part is of the form
>
> $$a_n = c_{n-1}a_{n-1} + c_{n-2}a_{n-2} + \cdots + c_{n-j}a_{n-j},\ c_i \in R$$
>
> is as follows:
>
> 1. Subtract the right member from both members, producing an equation of the form
>
> $$a_n - c_{n-1}a_{n-1} - c_{n-2}a_{n-2} - \cdots - c_{n-j}a_{n-j} = 0$$
>
>   Note that the indices of a are in decreasing order.
> 2. Replace each a_i by x^i. This produces a polynomial for the form
>
> $$x^n - c_{n-1}x^{n-1} - c_{n-2}x^{n-2} - \cdots - c_{n-j}x^{n-j} = 0$$
>
>   Then replace n by j; this produces a polynomial with a constant term $-c_{n-j}$.
> 3. Solve the polynomial, obtaining the roots $\alpha_1, \alpha_2, \ldots, \alpha_j$.
> 4. The solution is of the form $a_n = C_1\alpha_1^n + C_2\alpha_2^n + \cdots + C_j\alpha_j^n$. The values of the C_i are found by using the known values of $a_0, a_1, \ldots$ to obtain a system of equations.

As an example, consider the series in part 3 of example 12–7 A.

$$a_n = \begin{cases} 1 \text{ if } n = 0,\ 3 \text{ if } n = 1 \\ 2a_{n-1} + 3a_{n-2} \text{ if } n > 1 \end{cases} \qquad = 1, 3, 9, 27, 81, \ldots$$

The terms of the sequence appear to be a geometric sequence with $a_n = 3^n$. However, we cannot prove this directly from the definition.[25] Instead, we use recurrence relations, as follows.

	$a_n = 2a_{n-1} + 3a_{n-2}$	Recursive part of definition
Step 1:	$a_n - 2a_{n-1} - 3a_{n-2} = 0$	Arrange so all nonzero terms are on one member with subscripts in decreasing order
Step 2:	$x^n - 2x^{n-1} - 3x^{n-2} = 0$	Replace each a_i by x^i
	$x^2 - 2x^{2-1} - 3x^{2-2} = 0$	Replace n by 2
	$x^2 - 2x - 3 = 0$	
Step 3:	$(x - 3)(x + 1) = 0$	Factor
	$x = 3$ or -1	

[25]This could be proved by the method of finite induction (section 12–4). See problems 23 and 24 in the exercises.

Step 4: $a_n = A(3)^n + B(-1)^n$

a_n is a linear combination of the solutions, each to the power n. We can find the values of A and B by using a_0 and a_1.

$a_n = A(3)^n + B(-1)^n$

$n = 0$: $a_0 = A(3)^0 + B(-1)^0$ so $1 = A + B$

$n = 1$: $a_1 = A(3)^1 + B(-1)^1$ so $3 = 3A - B$

We solve this system of two linear equations in two variables using any of the methods of chapter 10, or substitution (section 3–2). We find that $A = 1$, $B = 0$. Thus, $a_n = (1)(3^n) + 0(-1)^n$, or $a_n = 3^n$.

Example 12–7 C illustrates further.

■ ***Example 12–7 C***

Find an expression for a_n for the sequences in parts 4 and 5 of example 12–7 A.

1. $a_n = \begin{cases} 2 \text{ if } n = 0, 3 \text{ if } n = 1 \\ 2a_{n-1} + 3a_{n-2} \text{ if } n > 1 \end{cases}$

$a_n = 2a_{n-1} + 3a_{n-2}$

$a_n - 2a_{n-1} - 3a_{n-2} = 0$

$x^n - 2x^{n-1} - 3x^{n-2} = 0$

$x^2 - 2x - 3 = 0$

$x = 3$ or -1

$a_n = A3^n + B(-1)^n$

$n = 0$: $a_0 = 2 = A(3^0) + B(-1)^0$ so $2 = A + B$

$n = 1$: $a_1 = 3 = A(3^1) + B(-1)^0$ so $3 = 3A - B$

$A = \frac{5}{4}, B = \frac{3}{4}$

$a_n = \frac{5}{4}(3^n) + \frac{3}{4}(-1)^n$, or $a_n = \frac{1}{4}[5(3^n) + 3(-1)^n]$.

2. $a_n = \begin{cases} 1 \text{ if } n = 0, 1 \text{ if } n = 1 \\ a_{n-1} + a_{n-2} \text{ if } n > 1 \end{cases}$

$a_n = a_{n-1} + a_{n-2}$

$a_n - a_{n-1} - a_{n-2} = 0$

$x^n - x^{n-1} - x^{n-2} = 0$

$x^2 - x - 1 = 0$

$x = \dfrac{1 + \sqrt{5}}{2}$ or $\dfrac{1 - \sqrt{5}}{2}$

$a_n = A\left(\dfrac{1 + \sqrt{5}}{2}\right)^n + B\left(\dfrac{1 - \sqrt{5}}{2}\right)^n$

Now find A and B using $a_0 = 1$ and $a_1 = 1$.

$a_n = A\left(\dfrac{1 + \sqrt{5}}{2}\right)^n + B\left(\dfrac{1 - \sqrt{5}}{2}\right)^n$

$n = 0$: $a_0 = 1 = A\left(\dfrac{1 + \sqrt{5}}{2}\right)^0 + B\left(\dfrac{1 - \sqrt{5}}{2}\right)^0$ so $1 = A + B$

$n = 1$: $a_1 = 1 = A\left(\dfrac{1 + \sqrt{5}}{2}\right)^1 + B\left(\dfrac{1 - \sqrt{5}}{2}\right)^1$

We now solve for A and B using substitution (section 3–2).

$$1 = A + B \quad \text{First equation from above}$$
$$B = 1 - A \quad \text{Solve for } B$$
$$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \quad \text{Second equation from above}$$
$$2 = A(1+\sqrt{5}) + B(1-\sqrt{5}) \quad \text{Multiply each term by 2}$$
$$2 = A(1+\sqrt{5}) + (1-A)(1-\sqrt{5}) \quad \text{Replace } B \text{ by } 1 - A$$
$$\sqrt{5} + 1 = 2A\sqrt{5} \quad \text{Add } \sqrt{5} \text{ to both members}$$
$$\frac{\sqrt{5}+1}{2\sqrt{5}} = A \quad \text{Divide both members by } 2\sqrt{5}$$

$$B = 1 - A = \frac{2\sqrt{5}}{2\sqrt{5}} - \frac{\sqrt{5}+1}{2\sqrt{5}} = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

Thus, $a_n = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$ is the general term of the Fibonacci sequence.

Roots of multiplicity greater than one

When we solve the polynomial equation that is part of the preceding procedure we may find roots that are of multiplicity greater than 1.

Roots of multiplicity $m > 1$

Roots of multiplicity m, $m > 1$, are treated by using powers of n from 0 to $m - 1$ as additional coefficients with the appropriate roots. For example, if γ (gamma) is a root of multiplicity 3, then the expression for a_n would include

$$c_i\gamma^n + nc_{i+1}\gamma^n + n^2c_{i+2}\gamma^n$$

This is illustrated in example 12–7 D.

■ ***Example 12–7 D***

Find an expression for the nth term of the sequence.

1. $a_n = \begin{cases} 1 \text{ if } n = 0,\ 3 \text{ if } n = 1 \\ 4a_{n-1} - 4a_{n-2} \text{ if } n > 1 \end{cases}$

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$
$$x^2 - 4x + 4 = 0$$
$$(x-2)^2 = 0$$
$x = 2$ is a root of multiplicity 2.
$$a_n = A(2^n) + Bn(2^n)$$
$$n = 0\colon a_0 = 1 = A$$
$$n = 1\colon a_1 = 3 = 2A + 2B$$
$$A = 1,\ B = \tfrac{1}{2}$$

$$a_n = 2^n + \frac{n}{2}(2^n), \text{ or } a_n = 2^n\left(\frac{n+2}{2}\right).$$

2. $a_n = \begin{cases} 1 \text{ if } n = 0, 3 \text{ if } n = 1, 4 \text{ if } n = 2 \\ a_{n-1} + a_{n-2} - a_{n-3} \text{ if } n > 2 \end{cases}$

$$a_n = a_{n-1} + a_{n-2} - a_{n-3}$$
$$a_n - a_{n-1} - a_{n-2} + a_{n-3} = 0$$
$$x^n - x^{n-1} - x^{n-2} + x^{n-3} = 0$$
$$x^3 - x^2 - x + 1 = 0$$
$$x^2(x-1) - 1(x-1) = 0$$
$$(x-1)(x^2-1) = 0$$
$$(x-1)(x+1)(x-1) = 0$$

$x = 1$ and -1 are zeros. 1 has multiplicity 2.

$$a_n = A(1^n) + Bn(1^n) + C(-1)^n$$
$$a_n = A + nB + (-1)^nC \quad \text{General form of solution}$$

$n = 0$: $a_0 = A + 0B + (-1)^0C$, so [1] $\quad 1 = A + C$

$n = 1$: $a_1 = A + B + (-1)^1C$, so [2] $\quad 3 = A + B - C$

$n = 2$: $a_2 = A + 2B + (-1)^2C$, so [3] $\quad 4 = A + 2B + C$

Solving shows that $A = \frac{5}{4}$, $B = \frac{3}{2}$, $C = -\frac{1}{4}$.

$$a_n = \tfrac{5}{4} + n(\tfrac{3}{2}) + (-1)^n(-\tfrac{1}{4})$$
$$= \frac{5 + 6n - (-1)^n}{4}$$

Mastery points

Can you

- Compute terms in recursively defined sequences?
- Find an expression for a_n in certain recursively defined sequences when that sequence is an arithmetic or geometric sequence?
- Find an expression for a_n in certain recursively defined sequences using recursion relations?

Exercise 12–7

Find the first five terms of each recursively defined sequence. Then find an expression for a_n.

1. $a_n = \begin{cases} 3 \text{ if } n = 0 \\ a_{n-1} + 5 \text{ if } n > 0 \end{cases}$

2. $a_n = \begin{cases} -10 \text{ if } n = 0 \\ a_{n-1} + 3 \text{ if } n > 0 \end{cases}$

3. $a_n = \begin{cases} 5 \text{ if } n = 0 \\ 2a_{n-1} \text{ if } n > 0 \end{cases}$

4. $a_n = \begin{cases} 2 \text{ if } n = 0 \\ -3a_{n-1} \text{ if } n > 0 \end{cases}$

5. $a_n = \begin{cases} -2 \text{ if } n = 0, 3 \text{ if } n = 1 \\ 2a_{n-1} + 3a_{n-2} \text{ if } n > 1 \end{cases}$

6. $a_n = \begin{cases} 2 \text{ if } n = 0, 3 \text{ if } n = 1 \\ 5a_{n-1} - 4a_{n-2} \text{ if } n > 1 \end{cases}$

7. $a_n = \begin{cases} 3 \text{ if } n = 0, 1 \text{ if } n = 1 \\ 3a_{n-1} + 4a_{n-2} \text{ if } n > 1 \end{cases}$

8. $a_n = \begin{cases} -2 \text{ if } n = 0, 4 \text{ if } n = 1 \\ 5a_{n-1} - 6a_{n-2} \text{ if } n > 1 \end{cases}$

9. $a_n = \begin{cases} -2 \text{ if } n = 0, 4 \text{ if } n = 1 \\ 3a_{n-1} + 6a_{n-2} \text{ if } n > 1 \end{cases}$

10. $a_n = \begin{cases} -2 \text{ if } n = 0, 4 \text{ if } n = 1 \\ 5a_{n-1} - a_{n-2} \text{ if } n > 1 \end{cases}$

Find an expression for the nth term of the sequence.

11. $a_n = \begin{cases} 1 \text{ if } n = 0, 3 \text{ if } n = 1 \\ 6a_{n-1} - 9a_{n-2} \text{ if } n > 1 \end{cases}$

12. $a_n = \begin{cases} 2 \text{ if } n = 0, 3 \text{ if } n = 1 \\ -6a_{n-1} - 9a_{n-2} \text{ if } n > 1 \end{cases}$

13. $a_n = \begin{cases} 4 \text{ if } n = 0, 3 \text{ if } n = 1 \\ 2a_{n-1} - a_{n-2} \text{ if } n > 1 \end{cases}$

14. $a_n = \begin{cases} -2 \text{ if } n = 0, 2 \text{ if } n = 1 \\ 8a_{n-1} - 16a_{n-2} \text{ if } n > 1 \end{cases}$

15. $a_n = \begin{cases} 1 \text{ if } n = 0, 1 \text{ if } n = 1, 3 \text{ if } n = 2 \\ a_{n-1} + a_{n-2} - a_{n-3} \text{ if } n > 2 \end{cases}$

16. $a_n = \begin{cases} 1 \text{ if } n = 0, 1 \text{ if } n = 1, 3 \text{ if } n = 2 \\ 6a_{n-1} - 12a_{n-2} + 8a_{n-3} \text{ if } n > 2 \end{cases}$

17. State a weakness of recursive definitions that a closed form expression for the definition does not have.

18. Show that the general expression for the Fibonacci sequence (example 12–7 C) can be transformed into $a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$.

19. In the text we stated that the sequence $a_n = \begin{cases} 2 \text{ if } n = 0 \\ a_{n-1} + 3 \text{ if } n > 0 \end{cases}$ is an arithmetic sequence. We did not prove this, but only observed this. Using the definition of arithmetic sequence prove that this sequence is indeed an arithmetic sequence.

20. In the text we observed, without proof, that the sequence $a_n = \begin{cases} 1 \text{ if } n = 0 \\ 3a_{n-1} \text{ if } n > 0 \end{cases}$ is a geometric sequence. Using the definition of geometric sequence prove that this sequence is indeed a geometric sequence.

21. In the text the sequence $a_n = \begin{cases} 1 \text{ if } n = 0, 3 \text{ if } n = 1 \\ 2a_{n-1} + 3a_{n-2} \text{ if } n > 1 \end{cases}$ was seen to be a geometric sequence, whereas the sequence $a_n = \begin{cases} 2 \text{ if } n = 0, 3 \text{ if } n = 1 \\ 2a_{n-1} + 3a_{n-2} \text{ if } n > 1 \end{cases}$ was not a geometric sequence, even though in both cases $a_n = 2a_{n-1} + 3a_{n-2}$ for $n > 1$. Find the value A so that $a_n = \begin{cases} 2 \text{ if } n = 0, A \text{ if } n = 1 \\ 2a_{n-1} + 3a_{n-2} \text{ if } n > 1 \end{cases}$ is a geometric sequence.

(*Hint:* For a sequence to be geometric $\frac{a_1}{a_0} = r$ and $\frac{a_2}{a_1} = r$ must be true for some $r > 0$, $r \neq 1$.)

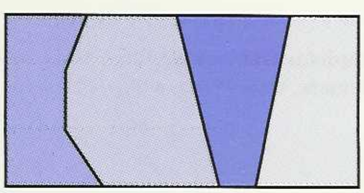
22. Given $a_n = \begin{cases} A \text{ if } n = 0, B \text{ if } n = 1 \\ 5a_{n-1} + 6a_{n-2} \text{ if } n > 1 \end{cases}$, find values of A and B so that the sequence is a geometric sequence. (See problem 21 for a hint.)

23. Use finite induction (section 12–4) to prove that $b_n = 3^n$ defines the same sequence as $a_n = \begin{cases} 1 \text{ if } n = 0, 3 \text{ if } n = 1 \\ 2a_{n-1} + 3a_{n-2} \text{ if } n > 1 \end{cases}$. That is, show that for all $n \in N$, each element a_n is of the form 3^n. (*Hint:* Show the cases for $n = 0$ or $n = 1$ by hand. Then, assume that $a_k = 2a_{k-1} + 3a_{k-2}$ is of the form 3^k for all natural numbers up to k, and use this to show that $a_{k+1} = 2a_k + 3a_{k-1}$ is of the form 3^{k+1}. Note that if a_k is of the form 3^k, then $a_{k-1} = 3^{k-1}$ and $a_{k-2} = 3^{k-2}$, since the statement is true for all values $0, 1, \ldots, k$.)

24. Use finite induction to prove that $b_n = 2n + 1$ defines the same sequence as $a_n = \begin{cases} 1 \text{ if } n = 0, 3 \text{ if } n = 1 \\ 2a_{n-1} - a_{n-2} \text{ if } n > 1 \end{cases}$ (See problem 23.)

Skill and review

1. The first term in an arithmetic progression is 4, and the 58th term is 67. Find the 96th term.
2. An artist is going to fill in each of the four sections of the work illustrated here.



   a. If the artist has 4 colors, and wants each section to be a different color, how many ways can the artist color the painting?
   b. If the artist has 6 colors, and wants each section to be a different color, how many ways can the artist color the painting?
   c. If the artist has 5 colors, and if the artist is willing to repeat colors in sections (including adjacent sections), how many ways can the artist color the painting?
3. Solve the inequality $x^2 - 10x - 18 > 6$.
4. Add up the numbers $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}$.

Chapter 12 summary

- **Arithmetic sequence** $a_{n+1} = a_n + d$

  $a_n = a_1 + (n-1)d$

  $S_n = \frac{n}{2}(a_1 + a_n)$

  $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

- **Geometric sequence** $a_{n+1} = r \cdot a_n$

  $a_n = a_1 r^{n-1}$

  $S_n = \frac{a_1(1 - r^n)}{1 - r}$

  $S_\infty = \frac{a_1}{1 - r}, \ |r| < 1$

- **Pascal's triangle**

```
            1   1
          1   2   1
        1   3   3   1
      1   4   6   4   1
    1   5   10  10  5   1
             etc.
```

- $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$
- ${}_nC_k = \frac{n!}{(n-k)!\,k!}$ if $n \geq k \geq 0$, $k, n \,\varepsilon\, N$.
- [1] $\binom{n}{n} = 1$ [2] $\binom{n}{1} = n$ [3] $\binom{n}{0} = 1$
- ${}_nP_r = n(n-1)(n-2) \cdots (n - [r-1])$
- **Binomial expansion formula** $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$
- Let k be a constant, and let $f(i)$ and $g(i)$ represent expressions in the index variable i. Then the following are true.

Sum of constants property:	$\sum_{i=1}^{n} k = nk$
Constant factor property:	$\sum_{i=1}^{n} [k \cdot f(i)] = k \cdot \sum_{i=1}^{n} f(i)$
Sum of terms property:	$\sum_{i=1}^{n} [f(i) + g(i)] = \sum_{i=1}^{n} f(i) + \sum_{i=1}^{n} g(i)$
Sum of integer series:	$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
Sum of squares of integers series:	$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
Sum of cubes of integers series:	$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

- **Principle of finite induction** If (1) a statement is true for $n = 1$, and
  (2) it can be shown that if the statement is true for $n = k$ then it must also be true for $n = k + 1$, then the statement is true for any positive integer.
- **Multiplication-of-choices property** If a choice in which the order in which each choice is made is not important consists of k decisions, where the first can be made n_1 ways and for each of these choices the second can be made in n_2 ways, and in general the ith choice can be made in n_i ways, then the complete choice can be made in $n_1 \cdot n_2 \cdot \cdots \cdot n_k$ ways. Each complete choice is also called an outcome.
- **Indistinguishable permutations** The number of distinct permutations P of n elements, where n_1 are alike of one kind, n_2 are alike of another kind, . . . , and n_k are alike of another kind, where $n_1 + n_2 + \cdots + n_k = n$, is $P = \frac{n!}{n_1!n_2! \cdots n_k!}$.
- **The probability of an event** If an event A is made up of $n(A)$ equally likely outcomes from a sample space S that has $n(S)$ equally likely outcomes, then the probability of the event A, represented by $P(A)$, is $P(A) = \frac{n(A)}{n(S)}$.
- **Basic probability principles** If the event A has $n(A)$ equally likely outcomes and the sample space S has $n(S)$ equally likely outcomes, and $n(S) > 0$, then
  1. $0 \leq P(A) \leq 1$.
  2. $P(A) = \frac{n(A)}{n(S)}$.
  3. $P(A) = 0$ means event A cannot occur, and is called an impossible event.
  4. $P(A) = 1$ means event A must occur and is called a certain event.
- **Probability of mutually exclusive events** If the events A and B are mutually exclusive, then the probability that A or B will occur is

$$P(A \text{ or } B) = P(A) + P(B)$$

- **General addition rule of probability** If the events A and B are from the same sample space, then the probability that A or B will occur is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

- **Probability of complementary events** If A is any event and A' is its complement, then $P(A') = 1 - P(A)$.

Chapter 12 review

[12–1] List the first four terms of each sequence.

1. $a_n = 6n - 2$ **2.** $b_n = n^2 - \dfrac{1}{n}$ **3.** $a_n = (n-1)^2$

Find an expression for the general term of each sequence.

4. 3,7,11,15, . . . **5.** −200,−160,−120, . . .
6. $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$

7. A traffic engineer measured the average number of cars passing through a certain intersection every 15 minutes and obtained the values 300, 350, 425, 525, ... What might the engineer expect for the next measurement?

Characterize the sequence as arithmetic, geometric, or neither. State the common difference or common ratio as appropriate.

8. 2, 8, 32, 128, . . . **9.** 2, 8, 16, 26, . . .
10. 2, 8, 14, 20, . . .

11. Find the number of terms in the arithmetic sequence 150, 148, 146, . . . , 118.

12. Suppose a_n and b_n are two arithmetic sequences, and a new sequence c is defined such that $c_n = a_n + 2b_n$. Is the new sequence an arithmetic sequence? Prove or disprove this statement.

Find a_n for the following geometric sequences for the given values of a_1, r, and n.

13. $a_1 = 1{,}024$, $r = -\frac{1}{2}$, $n = 5$ **14.** $a_1 = \frac{1}{40}$, $r = 2$, $n = 7$
15. $a_1 = 1$, $r = 0.1$, $n = 3$

16. Given a geometric sequence in which $a_1 = 81$ and $a_6 = \frac{1}{3}$. Find a_4.

17. Given a geometric sequence in which $a_3 = \frac{5}{4}$ and $a_5 = 5$ find a_6. Assume $r > 0$.

18. Suppose a_n and b_n are two geometric sequences, and a new sequence c is defined such that $c_n = (a_n)(2b_n)$. Is the new sequence a geometric sequence? Prove or disprove this statement.

19. A ball is dropped from a height of 12 feet. If the ball rebounds two-thirds of the height of its previous fall with each bounce, how high does it rebound on the
a. second bounce? **b.** fourth bounce? **c.** nth bounce?

[12–2] Expand the following sigma expressions.

20. $\sum_{j=1}^{4} (2j + 3)$ **21.** $\sum_{j=1}^{5} \dfrac{2j+1}{j+1}$

22. $\sum_{j=1}^{4} (-1)^j (j+1)^2$ **23.** $\sum_{j=1}^{3} \left(\sum_{k=1}^{j} (k - 1) \right)$

Find the sum of the series determined by the given arithmetic sequence.

24. 3, 9, 15, . . . , 99
25. −10, −14, −18, . . . , −66
26. $-8, -6\frac{1}{2}, -5, \ldots, 2\frac{1}{2}$
27. $a_1 = -\frac{3}{4}$, $d = \frac{1}{4}$; find S_{32}

Find the required nth partial sum for each geometric sequence.

28. $a_1 = \frac{2}{3}$, $r = 3$; find S_5 **29.** $\frac{4}{3}, -\frac{8}{9}, \ldots$; find S_5

30. $\sum_{k=1}^{6} \frac{1}{16}(2^k)$ **31.** $\sum_{k=1}^{10} (-3)^k$

Find the sum of the given infinite geometric series. If the sum of the series is not defined state that.

32. $\sum_{i=1}^{\infty} (\frac{1}{3})^i$ **33.** $\sum_{i=1}^{\infty} 4(\frac{1}{2})^i$

34. $3 - 2 + \frac{4}{3} - \cdots$

Find the rational number form of the repeating decimal number.

35. $0.323\overline{232}$ **36.** $0.312312\overline{312}$ **37.** $0.32\overline{22}$

38. A ball is dropped from a height of 18 meters. Each time it strikes the floor the ball rebounds to a height that is 60% of the previous height. Find the total distance that the ball travels before it comes to rest on the floor.

39. A biologist in a laboratory estimates that a culture of bacteria is growing by 15% per hour. How long will it be, to the nearest hour, before the population doubles?

[12–3] Expand and simplify each expression.

40. $\binom{12}{3}$ **41.** $\binom{21}{18}$ **42.** $\binom{n+2}{n-1}$

Expand and simplify the following expressions using the binomial expansion formula.

43. $(2x - y)^4$ **44.** $(a^2b - 3)^5$
45. Find the 14th term of $(5a^2 + b^5)^{16}$.

Compute the sum of the following series.

46. $\sum_{i=1}^{25} (i + 3)$ **47.** $\sum_{i=1}^{10} (4i^2 - 1)$ **48.** $\sum_{i=1}^{5} [i^2 - (\frac{1}{3})^i]$

49. Find a general expression for $\sum_{i=1}^{k} (i^2 - i + 1)$.

50. Show that $\binom{n}{0} = 1$.

[12–4] Prove that the following statements are true for all $n \in N$ using finite induction.

51. $4 + 7 + 10 + \cdots + (3n + 1) = \dfrac{n(3n + 5)}{2}$

52. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

53. Show that $n^3 - n$ is divisible by 3 for any natural number n.

[12–5]

54. There are three prizes in a box, A, B, and C.
 a. Draw a tree diagram of the ways in which the prizes can be drawn from the box.
 b. List all possible ways in which the prizes can be drawn (for example, CBA).

55. An individual has four horses and six saddles. How many different combinations of horses and saddles can this person ride?

56. A certain school offers ten courses. In how many different ways can someone take two of them, one after the other?

57. In how many different ways can a student answer all the questions on a quiz consisting of eight multiple-choice questions, where each question offers three choices?

58. From a standard deck of playing cards, in how many ways can a person select two hearts and a diamond?

59. Compute ${}_{10}P_2$.

60. In an eight-horse race, how many different first-second-third place finishes are possible?

61. A president and vice-president are to be elected from a club with 12 members. In how many different ways can these offices be filled?

62. Twenty people bought chances on a raffle in which there are first, second, third, and fourth prizes. No person can win two prizes. How many ways can these prizes be awarded?

63. How many different words can be formed using all the letters of the word "madam"?

64. Compute ${}_{18}C_{16}$.

65. Show that ${}_nC_k = {}_nC_{n-k}$.

66. On an examination consisting of 12 essay questions the student may omit any three. In how many different ways can the student select the problems to be answered (the order is not important)?

67. An artist has six pigments that can be added to a basic white paint to form other colors. How many colors can be formed using equal amounts of two pigments?

68. A shopper wants to make a salad using three fruits. If there are seven fruits available, how many salads are possible?

69. From a standard deck of playing cards how many ways can a person select two aces and three kings, without regard to the order of selection?

70. Suppose there are 15 players on a baseball team.
 a. In how many different ways can 12 players be sent to a charity event?
 b. In how many different ways can a team of 9 be chosen if every player can play every position?
 c. In how many different ways can a captain and a cocaptain be chosen (assuming the cocaptain and captain are different positions)?
 d. How many different batting orders are possible (considering all possible 9-player teams and all possible batting orders for 9 players)?

71. **a.** How many ways can eight individuals, who happen to be four females and four males, sit in a row?
 b. How many ways can four females and four males sit in a row if females and males must alternate?
 c. How many arrangements of females and males sitting in a row are possible (i.e., permutations in which females are indistinguishable and males are indistinguishable)?

72. In how many different ways can a student answer eight of the questions on a quiz consisting of ten multiple-choice questions, if each question offers three choices?

73. Selecting from the set of digits $\{1, 2, 3, 4, 5, 6\}$ (repeat selections are allowed) how many of the following are possible?
 a. Four-digit numbers
 b. Four-digit odd numbers
 c. Three-digit numbers where the first digit must be even
 d. Three-digit numbers using only even digits.

74. Answer problem 73 if repetition of a digit is not allowed.

75. Eight teams are in a bowling league. If each team is required to play every other team twice during the season, what is the total number of league games that will be played?

76. A restaurant has four appetizers, six main dishes, and eight desserts. For a fixed price two diners at the same table can choose two from each of these categories. In how many ways can this choice be made?

77. If a group consists of 10 men and 12 women, in how many different ways can a committee of six be selected if:
   a. The committee is to have an equal number of men and women?
   b. The committee is to be all the same sex?
   c. There are no restrictions on membership on the committee?

78. A test contains three groups of questions, A, B, and C, which contain eight, four, and six questions, respectively. If a student must select five questions from group A and three from each of the remaining groups, how many different tests are possible?

[12–6] A coin is tossed four times. Find the probability of

79. one head and three tails.
80. no tails.

A card is drawn from a standard deck of playing cards. What is the probability of

81. a five.
82. a diamond.
83. a card from four through ten, inclusive.
84. a black king.
85. a diamond or a jack.
86. a diamond or a face card.
87. a card that is not a club.

A bowl contains 14 balls. Four are red, 4 are blue, and 6 are white. If one ball is randomly selected, what is the probability that the ball is

88. red?
89. red or blue?
90. not blue?

In a nursery there are 12 rubber plants. Four of the plants are diseased, although this is not detectable. If 6 of the plants are selected for a delivery, what is the probability that

91. two of the plants are diseased?
92. a. all of the diseased plants are in the shipment?
   b. none of the diseased plants are in the shipment?

Five cards are drawn from a standard deck of playing cards. Find the probability of each event.

93. All five cards are clubs.
94. All five cards are red.
95. None of the cards are red.
96. Two black and three red cards.

97. In a certain state lottery six numbers must be chosen correctly from 49 numbers. What is the probability of making this choice correctly?

98. In a nursery there are 12 rubber plants. Four of the plants are diseased, although this is not detectable. If a customer buys 1 of the 12 plants, what is the probability that the customer gets a diseased plant?

[12–7] Find the first five terms of each recursively defined sequence. Then find an expression for a_n.

99. $a_n = \begin{cases} 2 \text{ if } n = 0 \\ a_{n-1} + 6 \text{ if } n > 0 \end{cases}$

100. $a_n = \begin{cases} 3 \text{ if } n = 0 \\ 2a_{n-1} \text{ if } n > 0 \end{cases}$

101. $a_n = \begin{cases} 2 \text{ if } n = 0,\ 3 \text{ if } n = 1 \\ 2a_{n-1} + a_{n-2} \text{ if } n > 1 \end{cases}$

102. $a_n = \begin{cases} 2 \text{ if } n = 0,\ 3 \text{ if } n = 1 \\ 2a_{n-1} - a_{n-2} \text{ if } n > 1 \end{cases}$

103. $a_n = \begin{cases} 1 \text{ if } n = 0,\ 3 \text{ if } n = 1 \\ 4a_{n-1} - 4a_{n-2} \text{ if } n > 1 \end{cases}$

Chapter 12 test

List the first four terms of each sequence.

1. $a_n = (-1)^{n+1}(-n + 3)$
2. $b_n = n - \dfrac{1}{n}$

Find an expression for the general term of each sequence.

3. 6, 10, 14, 18, . . .
4. $3, 2, \frac{5}{3}, \frac{3}{2}, \frac{7}{5}, \ldots$

5. Find the number of terms in the arithmetic sequence 103, 106, 109, . . . , 184.

6. Find a_5 for the arithmetic sequence in which $a_7 = 12$ and $a_{13} = 28$.

7. Suppose A is an arithmetic sequence, and a new sequence B is defined such that $b_n = 3a_n$. Is sequence B an arithmetic sequence? Prove or disprove this statement.

8. Find a_4 for the geometric sequence in which $a_1 = 243$ and $r = -\frac{2}{3}$.

9. Given a geometric sequence in which $a_3 = 200$ and $a_5 = 25$, find a_7.

10. Suppose A is a geometric sequence, and a new sequence B is defined such that $b_n = a_n + 1$. Is the new sequence a geometric sequence? Prove or disprove this statement.

11. A ball is dropped from a height of 36 feet. If the ball rebounds two-thirds of the height of its previous fall with each bounce, how high does it rebound on the

a. second bounce? **b.** fourth bounce? **c.** nth bounce?

Expand the following sigma expressions.

12. $\sum_{j=1}^{4} \frac{2j}{j^2 + 1}$ **13.** $\sum_{i=1}^{5} (-1)^i(i - 3)^2$

14. $\sum_{j=1}^{3} \left(\sum_{k=1}^{j} k^2\right)$

Find the sum of the series determined by the given arithmetic sequence.

15. $-20, -18, -16, \ldots, 18, 20, 22$

16. $a_1 = 80, d = \frac{1}{4}$; find S_{32}

Find the required nth partial sum for each geometric sequence.

17. $a_1 = \frac{1}{3}, r = 6$; find S_5. **18.** $6, -1, \ldots$; find S_5.

19. $\sum_{k=1}^{8} 512(-\frac{1}{2})^k$ **20.** $\sum_{k=1}^{5} (0.3)^k$

Find the sum of the given infinite geometric series. If the sum of the series is not defined state that.

21. $\sum_{i=1}^{\infty} 3^i$ **22.** $\sum_{i=1}^{\infty} 27(\frac{1}{3})^i$

Find the rational number form of the repeating decimal number.

23. $0.2727\overline{27}$ **24.** $0.433\overline{3}$

25. A ball is dropped from a height of 40 meters. Each time it strikes the floor the ball rebounds to a height that is 75% of the previous height. Find the total distance that the ball travels before it comes to rest on the floor.

26. An investment grows at a rate of 10% per year. How long will it be, to the nearest tenth of a year, before the investment doubles?

Expand and simplify each expression.

27. $\binom{8}{4}$ **28.** $\binom{n}{n-1}$

Expand and simplify the following expressions using the binomial expansion formula.

29. $(x^2 - 3y)^4$

30. Find the 11th term of $(3a^2 + b^5)^{14}$.

Compute the sum of the following series.

31. $\sum_{i=1}^{14} (i + 3)^2$ **32.** $\sum_{i=1}^{5} [i - (\frac{1}{4})^i]$

33. Find a general expression for $\sum_{i=1}^{k} (2i + 1)$.

34. If $\binom{n}{2} = 66$, find n.

Prove that the following statements are true for all $n \varepsilon N$ using finite induction.

35. $5 + 9 + 13 + \cdots + (4n + 1) = 2n^2 + 3n$

36. $3 + \frac{3}{2} + \frac{3}{4} + \cdots + \frac{3}{2^{n-1}} = \frac{6(2^n - 1)}{2^n}$

37. Show that $n^2 + 7n + 12$ is divisible by 2 for any natural number n.

38. A traveler wishes to visit France (F), England (E), and Germany (G).

a. Draw a tree diagram of the ways in which the traveler can visit these three countries once, one after the other.

b. List all possible ways in which the countries can be visited (for example, FEG).

39. An individual has five pairs of shoes, four pairs of pants, and six shirts. How many different combinations of shoes, pants, and shirts can this person wear?

40. In how many different ways can a student answer all the questions on a quiz consisting of ten multiple-choice questions, where each question offers four choices?

41. From a standard deck of playing cards, in how many ways can a person select a hand consisting of two hearts and two clubs?

42. Compute ${}_{15}P_3$.

43. A background color and a foreground color are to be chosen from the color menu on a computer art program. There are 15 colors, and the background and foreground colors should be different. In how many different ways can these colors be chosen?

44. How many sentences can be formed using all the words in the sentence "to be or not to be, that is not a question," neglecting punctuation and meaning?

45. A computer has been programmed to print out all the different ways the 26 letters of the alphabet A, B, C, . . . , Z can be written down with no repetition, using all 26 letters.

a. How many ways is this?

b. If the computer prints 10 of these ways per second, how long will it take to print out all of them?

46. Compute ${}_{30}C_{26}$.

47. Simplify ${}_nC_{n-2}$; the answer should not use factorial notation.

48. On an examination consisting of ten essay questions the student must answer any eight. In how many different ways can the student select the problems to be answered (the order is not important)?

49. A club of 23 students is to choose 2 to represent the club at a student government meeting. How many ways can these 2 students be chosen?

50. From an ordinary deck of playing cards how many ways can a person select two kings, three queens, and a jack, without regard to the order of selection?

Suppose there are 20 players on a baseball team in problems 51 through 54.

51. In how many different ways can a team of 9 be chosen if every player can play every position?

52. In how many different ways can a captain and a cocaptain be chosen (assuming the cocaptain and captain are different positions)?

53. How many different batting orders are possible (considering all possible 9-player teams and all possible batting orders for 9 players)?

54. How many different ways can a team of 9 be chosen if three of the players are pitchers and each team must have exactly one pitcher?

55. In how many different ways can a student answer six of the questions on a quiz consisting of ten multiple-choice questions, if each question offers four choices?

56. Using the ten digits 0, 1, . . . , 9, how many three-digit numbers can be formed if repetition of a digit is not allowed?

57. Twenty-four players enter a tennis tournament. Each player is required to play one match with every other player. What is the total number of matches that will be played?

58. In a modern European literature course students are offered a choice of what they must read from six countries. The number of reading choices are categorized as post-war P or contemporary C as follows:

Country	P	C	Country	P	C
France	3	2	Russia	3	2
Germany	2	2	Spain	3	1
Italy	1	3	United States	2	3

For example, for the United States, the student is offered two post-war writers and three current writers.

a. If a student must choose any two works from each country, how many ways can this selection be made?

b. If a student must choose one post-war and one current work from each country, how many ways can this be done?

59. A coin is tossed four times. Find the probability of two heads and two tails.

A card is drawn from a standard deck of playing cards. What is the probability of

60. a red five?

61. a red card or a face card?

62. not getting a 5?

63. A bowl contains 22 balls. Twelve are red, 4 are blue, and 6 are white. If 1 ball is randomly selected, what is the probability that the ball is not white?

64. An employee of a state weights and measures department selects a carton that contains 30 bags of potato chips; 4 of the bags are underweight. The employee selects 5 of the bags at random. What is the probability that at least 1 of the 5 bags selected is underweight? (Note that this event is the complement of the event in which none of the bags is underweight.)

65. In a radio contest, a caller will win if the caller selects two numbers between 1 and 20 inclusive correctly. What is the probability of winning?

66. Show that, for $r \geq 1$, $\dfrac{{}_nC_r}{{}_nC_{r-1}} = \dfrac{n-(r-1)}{r}$.

Find the first five terms of each recursively defined sequence. Then find an expression for a_n.

67. $a_n = \begin{cases} 5 \text{ if } n = 0 \\ a_{n-1} + 2 \text{ if } n > 0 \end{cases}$

68. $a_n = \begin{cases} 2 \text{ if } n = 0 \\ 3a_{n-1} \text{ if } n > 0 \end{cases}$

69. $a_n = \begin{cases} 2 \text{ if } n = 0,\ 4 \text{ if } n = 1 \\ 2a_{n-1} - a_{n-2} \text{ if } n > 1 \end{cases}$

70. $a_n = \begin{cases} 2 \text{ if } n = 0,\ 3 \text{ if } n = 1 \\ 3a_{n-1} - a_{n-2} \text{ if } n > 1 \end{cases}$